Measurement as the Closure of the Ladder: A Synthesis of the Modular Library *Mathematics* $\leftrightarrow$ *Physical Representation*

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Abstract

This is the synthesis rung of a standalone modular library whose governing thesis is that *a Goncharov-style Lie coalgebra is a bookkeeping system for how physical observables decompose into irreducible informational constituents.* The library is a ladder of five standalone parts, each adding exactly one emergent structural property the level below cannot express: Part I disciplines a *domain* of observable ratios (a field F, its multiplicative group $F^\times$, cross-ratios, the moduli spaces $\mathcal{M}_{0,n}$); Part II populates that domain with weight-graded *functions* (classical and multiple polylogarithms, iterated integrals), yielding the slogan transcendental weight = loop order; Part III recognises the transcendental constants of Part II as *periods* of a single pre-numerical *object*, the motive; Part IV endows that object with a coproduct Δ and cobracket δ — the *decomposition law* of observables, the Goncharov Lie coalgebra $\mathcal{L}_G(F)$; and Part V supplies the arrow that *evaluates* the object to a measurable real number, the period pairing $\mathrm{per}(\Pi) = \int_\Gamma \omega$ and the regulator. This paper recomposes the five parts. We give a single unified framework in which all five rungs are faces of one object — the connected graded Hopf algebra $\mathcal{H}(F)$ of motivic periods, its Lie coalgebra of indecomposables $\mathcal{L}_G(F) = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2$, and its period realisation — and we prove a *Closure Theorem*: every tame (mixed Tate) observable is the period pairing of a motivic object whose coalgebraic anatomy terminates at primitives $P_1 \cong F^\times \otimes \mathbb{Q}$ (Part I's ratios) together with irreducible higher-weight constants, and the real number it returns is *itself* a new element of a Part I domain, so the pipeline I $\to$ II $\to$ III $\to$ IV $\to$ V $\to$ I closes into a loop. The capstone emergent property, absent from every individual rung, is **measurement as the closure of the ladder**: the regulator/period pairing of Part V closes the loop opened by the domain of observables of Part I. We exhibit the five-term dilogarithm relation as the golden thread that is simultaneously kinematic (I), functional (II), motivic (III), a cobracket kernel (IV), and a regulator consistency condition (V); we harmonise and audit the library's three-tier epistemic status system $\mathsf{S}/\mathsf{H}/\mathsf{P}$; and we collect the

library's limitations and open problems. A compiling Haskell demonstration of the ladder composite and a self-contained Lean 4 sketch accompany the paper.

Contents

1 Introduction

1.1 The governing perspective

A physical observable — a scattering amplitude, a Feynman period, an entropy, a special value of an L-function — is treated throughout this library as the *numerical realisation* of a deeper, pre-numerical, structured object. The question "what is this observable made of?" is answered not by further computation but by a *decomposition law*: a coproduct or cobracket that exposes the observable's primitive, irreducible constituents (its cuts, discontinuities, and factorisation channels). The organising sentence of the whole library, reproduced verbatim in each of its parts, is:

> A Goncharov-style Lie coalgebra is not merely a formal algebraic object; it is a *bookkeeping system* for how physical quantities break apart into irreducible informational constituents.

The compressed slogans that recur across every part are

$$\text{observable} = \text{Real}(\text{motivic information}), \quad \text{coalgebra} = \text{grammar of decomposition},$$

"physics observes numerical shadows of deeper motivic structures," and its strongest (and most speculative) form, "physical reality is the realisation layer of structured mathematical information."

1.2 A modular, not unified, framework

The library is deliberately **modular**. It is not a single grand unified construction but a *ladder* of five standalone, individually defensible parts, each of which composes with its neighbour to add exactly one emergent structural property that the level below cannot express on its own. Table 1 records the ladder and the present synthesis that recomposes it.

We stress the word *modular*. There is no single universal functor $\mathsf{Math} \to \mathsf{Phys}$; each rung is a local, domain-indexed construction, and the value of the library lies precisely in keeping the rungs separable yet composable. The synthesis does not collapse the five parts into one; it exhibits the *composite* and the *emergent properties* that only the composite possesses.

1.3 What this paper does

This synthesis has five tasks, executed in Sections 3 to 7.

1. **Recompose the ladder explicitly** (Section 3): rung by rung, we recall each part's new structure, name the gap it closes in the level below, and prove or argue that the hand-off $\mathrm{I} \to \mathrm{II} \to \mathrm{III} \to \mathrm{IV} \to \mathrm{V}$ is a concrete mathematical fact, not a metaphor.

Part	New structure	Emergent property
I	Field F, group $F^\times$, cross-ratios, moduli $\mathrm{Conf}_n(\mathbb{P}^1) = \mathcal{M}_{0,n}$	A disciplined domain of observable values
II	Li_n, iterated integrals, weight/depth grading, the symbol	Transcendental depth = loop order
III	Motives $\mathrm{Mot}(X)$, realisations, comparison isomorphism, mixed Tate motives	The pre-numerical object behind measurement
IV	Hopf algebra $\mathcal{H}$, coproduct Δ, cobracket δ, $\mathcal{L}_G(F)$, primitives	The decomposition law of observables
V	Regulators, Deligne–Beilinson cohomology, period pairing $\int_\Gamma \omega$	Passage from motive to measurable number
Synth.	**Recomposition into observable** = per(**motive**)	**Measurement as the closure of the ladder**

Table 1: The modular ladder. Each part is standalone; composition creates the emergent property in the right-hand column. This paper is the synthesis rung.

2. **Give a unified mathematical framework** (Section 4): a single object — the connected graded Hopf algebra $\mathcal{H}(F)$ of motivic periods — of which all five rungs are faces. We prove a *Five Faces* theorem locating each part inside $\mathcal{H}(F)$.

3. **Prove the closure/capstone result** (Section 5): the *Closure Theorem* that the composite reproduces observable = per(motive) and that the output re-enters Part I's domain, so the ladder closes into a loop. Measurement is the arrow that closes it.

4. **Demonstrate emergent properties from composition** (Section 6): properties no single rung exhibits — the five-term relation read five ways, a single integer (weight) read three ways, and the cut/regulator complementarity.

5. **Audit the epistemic status** (Section 7): harmonise the S/H/P discipline across the five parts and tabulate its statistics, then collect limitations (Section 8) and open problems (Section 9).

1.4 Standalone-library discipline

This project is self-contained. All citations are to the external mathematical and physical literature listed in the references; no sibling project is cited, linked, or relied upon. The reader is assumed to have read, or to have to hand, Parts I–V of the library; we recall from each exactly what the synthesis needs and cite the part rather than reprove its internal results. References to the five parts are by roman numeral (*Part I*, ..., *Part V*); references to the external literature are numeric.

2 The Dictionary, the Status System, and the Master Formulas

Every part of the library opens with the same translation table, the same three-valued status system, and the same master formulas. We reproduce them here because the synthesis's job is to show that a *single* dictionary, applied consistently, threads all five rungs.

2.1 The mathematics $\leftrightarrow$ physics dictionary

Mathematics	**Physical representation**
Field F	Kinematic/coordinate domain of possible values
Element of $F^\times$	Ratio, cross-ratio, scale, observable parameter
Polylogarithm $\mathrm{Li}_n(x)$	Iterated propagation / layered amplitude contribution
Period	Measurable physical number obtained from geometry
Motive	Pre-numerical informational object behind the measurement
Lie algebra	Infinitesimal symmetry / generator structure
Lie coalgebra	Decomposition law of observables
Cobracket δ	Factorisation channel, cut, boundary, or information split
Weight grading	Complexity depth / loop order / transcendental depth
Primitive element	Irreducible informational unit
Coproduct	Full hierarchical decomposition
Regulator map	Passage from motivic object to physical real number
Hodge/de Rham realisation	Analytic form seen by calculation
Betti realisation	Topological/path/integration-cycle form
Period pairing	Physical observable as pairing of form and cycle

Label	Meaning	Canonical example
S	Standard use in mathematics or mathematical physics	The Goncharov coproduct on motivic polylogarithms
H	Strong heuristic representation-dictionary entry	Cobracket as "factorisation channel"; motive as "functional essence"
P	Speculative philosophical ontology	A universal cosmic Galois group acting on all QFT amplitudes; physical reality as motivic realisation

Table 3: The three-valued epistemic status system. Composite labels S/H and H/P occur where an entry is standard as pure mathematics but heuristic or speculative in its physical reading.

2.2 The epistemic status system, harmonised

The dictionary mixes rigorous mathematics with interpretive proposals of varying boldness. Every entry and every claim is tagged with one of three labels, following Table 3.

The five parts state the *composition rule* for these labels in two superficially different ways: Parts I, III, V phrase it as "the minimum under $\mathsf{S} < \mathsf{H} < \mathsf{P}$," while Parts II, IV phrase it as "the maximum." Both are glosses on the same intended content — *a chain of translations is only as reliable as its weakest link* — and it is a small but genuine task of the synthesis to state the rule unambiguously.

Principle 2.1 (Status monotonicity, harmonised; status S). Order the three labels by *increasing speculativeness / decreasing reliability*, $\mathsf{S} < \mathsf{H} < \mathsf{P}$. If two dictionary entries or two claims compose, the status of the composite is the *least reliable* of the two — equivalently, the *maximum* under the order $\mathsf{S} < \mathsf{H} < \mathsf{P}$. In particular a P-labelled step cannot be upgraded to S merely by composing it with otherwise-rigorous results, and no chain of compositions can raise the status of its weakest link. (The "minimum" phrasing of Parts I, III, V refers to the minimum *reliability*; the "maximum" phrasing of Parts II, IV to the maximum *speculativeness*. The two agree, and we adopt the reliability reading throughout.)

Theorem 2.1 is used silently below and explicitly in the epistemic audit of Section 7. Its force in the synthesis is sharp: the whole library's strongest structural results are S as mathematics, but the moment they are composed with the H physical reading "this motive is a physical observable," the composite inherits status H. The synthesis never presents a composite as more reliable than its weakest link.

$$
\begin{tikzcd}
F^{\times} \arrow[r, "\lambda"] \arrow[rrrrr, bend left, "\text{closure: per re-enters } F^{\times}"] & \mathrm{Li}_n(x) \arrow[r, "\mathrm{Mot}"] & \mathcal{A}^{\mathfrak{m}} \in \mathcal{H}(F) \arrow[r, "\Delta', \delta"] & \mathcal{L}_G(F) \arrow[r, "\mathrm{per}"] & \mathcal{A} \in \mathcal{P} \arrow[r, "\subset \mathbb{R}"] & F'^{\times} \\
(\mathrm{I}) & (\mathrm{II}) & (\mathrm{III}) & (\mathrm{IV}) & (\mathrm{V}) & (\mathrm{I})'
\end{tikzcd}
$$

Figure 1: The ladder as a composite that closes into a loop. Part I supplies the domain $F^{\times}$; Part II the functions Li_n; Part III the motivic object $\mathcal{A}^{\mathfrak{m}}$; Part IV its decomposition Δ', δ into the Lie coalgebra $\mathcal{L}_G(F)$; Part V the period map per producing a real number $\mathcal{A}$; and the curved arrow is the closure of Section 5: that number is itself an element of a new Part I domain $F'^{\times}$.

2.3 Master formulas

The following notation is fixed for the whole library. Each line is the responsibility of one rung, and the synthesis is the assertion that they compose.

$$
\begin{aligned}
\text{observable} &= \mathrm{Real}(\text{abstract structure}) && \text{(master slogan)} && (1)\\
\mathcal{A} &= \mathrm{per}(\mathcal{A}^{\mathfrak{m}}) && \text{(amplitude = period of motive)} && (2)\\
\Delta(\mathcal{A}^{\mathfrak{m}}) &= \sum\nolimits_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2} && \text{(coproduct / decomposition)} && (3)\\
\delta : \mathcal{L}_G(F) &\to \textstyle\bigwedge^2 \mathcal{L}_G(F) && \text{(cobracket / primitive split)} && (4)\\
\mathrm{wt}(x) = n \ \ (x \in \mathcal{L}_G(F)_n&), \ \ \mathrm{dep}(x) = \text{depth} && \text{(weight / depth gradings)} && (5)\\
\mathrm{per}(\Pi) &= \int_{\Gamma} \omega, \ \ \Pi = (X, D, \omega, \Gamma) && \text{(period pairing)} && (6)
\end{aligned}
$$

Here F is a field (a number field, or a field of kinematic invariants); $F^{\times}$ its multiplicative group; $\mathcal{A}^{\mathfrak{m}}$ is a *motivic amplitude object*, a pre-numerical element of a connected graded Hopf algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ of motivic periods, with $\mathrm{per}(\mathcal{A}^{\mathfrak{m}}) = \mathcal{A}$ its numerical realisation; and $\mathcal{L}_G(F) = \bigoplus_{n \geq 1} \mathcal{L}_G(F)_n = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2$ is the Goncharov Lie coalgebra of primitives, graded by weight n, with δ the antisymmetrisation of the reduced coproduct $\Delta'(x) := \Delta(x) - x \otimes 1 - 1 \otimes x$. Part I owns $F, F^{\times}$ in (2) and the weight-one piece of (4); Part II owns the grading (5) and the functions realised by per; Part III owns the object $\mathcal{A}^{\mathfrak{m}}$; Part IV owns (3)–(4); Part V owns (6). The synthesis owns the assertion that (1) is their composite.

3 The Five Rungs Recomposed

We recall each rung's contribution and, crucially, the concrete hand-off to the next. The reader should keep Figure 1 in view: it is the backbone of the whole synthesis.

3.1 Part I: the disciplined domain of observable values

Part I builds the ground floor. A physical process with n particles has a *kinematic domain*: the field F generated by its Lorentz invariants, whose multiplicative group $F^\times$ is the group of nonzero scales and dimensionless ratios (status $\mathsf{S/H}$). The *cross-ratio*

$$r(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)} \in F^\times \tag{7}$$

is the universal PGL_2-invariant of four marked points on $\mathbb{P}^1(F)$ (status S), and Part I's central theorem is that the cross-ratio identifies the four-point moduli space with the thrice-punctured line,

$$\mathcal{M}_{0,4} = \mathrm{Conf}_4(\mathbb{P}^1) \xrightarrow{\sim} \mathbb{P}^1 \setminus \{0, 1, \infty\}. \tag{8}$$

The gap in "the level below" that Part I closes is the *absence of a disciplined arena*: an amplitude is a function of *something*, and Part I says precisely of what — gauge-reduced, cross-ratio-coordinatised, boundary-compactified configurations. Its emergent property is *a disciplined domain of observable values.*

Remark 3.1 (The first hand-off, I → II; status $\mathsf{S/H}$). The right-hand side of (8) is *exactly* the domain on which the classical polylogarithm $\mathrm{Li}_n(z)$ of Part II is single-valued and holomorphic away from its branch points at $0, 1, \infty$. Part I therefore does not merely supply "some field"; it supplies the specific punctured domain that Part II's functions live on. The three punctures are the three ways two of the four points collide — geometrically the three boundary components of $\overline{\mathcal{M}_{0,4}}$, physically the three degeneration channels. This is the map λ of Figure 1.

3.2 Part II: transcendental weight as loop order

Part II populates the domain (8) with a weight-graded family of transcendental functions: the classical polylogarithm $\mathrm{Li}_n(x) = \sum_{k\geq 1} x^k/k^n$, the multiple polylogarithms $\mathrm{Li}_{n_1,\ldots,n_d}$, the multiple zeta values, and their common analytic incarnation as Chen iterated path integrals

$$\mathrm{Li}_n(x) = -I(0; 1, \underbrace{0, \ldots, 0}_{n-1}; x), \qquad I(a_0; a_1, \ldots, a_N; a_{N+1}) = \int_{0<t_1<\cdots<t_N<1} \prod_{i=1}^{N} \frac{\mathrm{d}t_i}{t_i - a_i}. \tag{9}$$

Two gradings appear: the *weight* $n = n_1 + \cdots + n_d$ (transcendental complexity) and the *depth* d (nesting). The emergent property is the slogan

$$\boxed{\text{transcendental weight} = \text{loop order}} \qquad \text{(status } \mathsf{S/H}\text{)}. \tag{10}$$

The gap Part II closes is the *absence of functions*: Part I gives a domain but says nothing about which functions live on it or how their complexity is measured. Part II's mathematical core — which makes (10) bounded and precise — is the termination of the reduced coproduct: for a weight-n polylogarithm the iterated Δ' strictly lowers weight in each factor and terminates after at most $n - 1$ steps at weight-one logarithmic leaves. Chen's theorem (iterated integrals of closed logarithmic forms are homotopy-functionals of the path) makes Li_n a well-defined multivalued function on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, whose monodromy is exactly the analytic-continuation data the later cobracket organises.

Remark 3.2 (The hand-off II $\to$ III; status S). The specific transcendental numbers Part II produces are not independent facts: the multiple polylogarithms are periods of the motivic fundamental group $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0,1,\infty\})$, and the multiple zeta values are their regularised values at $x = 1$. This is the map Mot of Figure 1: "functions on a domain" become "periods of an object."

3.3 Part III: the pre-numerical object behind measurement

Part III answers the recurrence puzzle — why the *same* constants (π, $\zeta(3)$, log's, polylogarithm values) recur across physically unrelated computations — with the theory of *motives*. The motive $\mathrm{Mot}(X)$ is the universal object through which every Weil cohomology theory of a variety X factors; its *realisations* (Betti, de Rham, Hodge, étale) are the shadows it casts, and a *period* is a matrix entry of the comparison isomorphism

$$\mathrm{comp}: \; H^k_{\mathrm{dR}}(X \setminus D/F) \otimes_F \mathbb{C} \xrightarrow{\sim} H^k_{\mathrm{B}}(X \setminus D, \mathbb{C}), \qquad \mathrm{per}(\Pi) = \langle \mathrm{comp}(\omega), \Gamma \rangle = \int_\Gamma \omega. \quad (11)$$

The gap Part III closes is the *brute-fact status* of Part II's constants: after Part III, "$\zeta(3)$ appears again" is the visible trace of a shared pre-numerical object, and the right question becomes "*which* motive is common?" Two anchors make the object real rather than conjectural: the Tannakian category of mixed realisations (after Huber), and the *unconditional* construction of the category $\mathrm{MTM}(F)$ of mixed Tate motives over a number field (Bloch–Kriz, Deligne–Goncharov, Levine; Brown over $\mathbb{Z}$). Its emergent property is exactly its title: the motive is the pre-numerical object behind measurement, and $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ of (2) first acquires content — the amplitude *is* the period of a genuine object $\mathcal{A}^{\mathfrak{m}}$.

Remark 3.3 (The hand-off III $\to$ IV; status S). The unconditional $\mathrm{MTM}(F)$ supplies the actually-constructed connected graded Hopf algebra $\mathcal{H}(F)$ of motivic periods on which Part IV's coproduct and cobracket act. Without Part III, Part IV's Δ would be a formal operation on an unconstructed object; with it, Δ acts on a theorem. The weight-one classifying datum $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q}$ (period $\log x$) is the bridge that identifies Part I's ratios with the weight-one primitives of Part IV.

3.4 Part IV: the decomposition law of observables (the core)

Part IV is the core. On the connected graded commutative Hopf algebra $\mathcal{H} = \mathcal{H}(F)$ it constructs the space of indecomposables $\mathcal{L}_G(F) = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2$ (with $\overline{\mathcal{H}} = \ker \varepsilon$ the augmentation ideal) and equips it with the *cobracket*

$$\delta: \; \mathcal{L}_G(F) \xleftarrow{q} \overline{\mathcal{H}} \xrightarrow{\Delta'} \overline{\mathcal{H}} \otimes \overline{\mathcal{H}} \xrightarrow{q \otimes q} \mathcal{L}_G(F) \otimes \mathcal{L}_G(F) \xrightarrow{\pi} \bigwedge^2 \mathcal{L}_G(F), \qquad (12)$$

the antisymmetrisation of the reduced coproduct. Part IV's technical heart is the *well-definedness theorem*: δ descends to indecomposables (the symmetric residue of $\Delta'(xy)$ is killed by the antisymmetriser π), lands in $\bigwedge^2$, and satisfies the co-Jacobi identity, so $(\mathcal{L}_G(F), \delta)$ is a graded Lie coalgebra (status S). Its structural anchor is the *cogeneration* *theorem*: the primitives $P = \bigoplus_n P_n$ (with $P_n = \ker \delta|_{\mathcal{L}_G(F)_n}$) cogenerate $\mathcal{L}_G(F)$, with a canonical weight-one isomorphism

$$P_1 = \mathcal{L}_G(F)_1 \;\cong\; F^\times \otimes \mathbb{Q}, \qquad x \longmapsto \{\text{Kummer class, period } \log x\}, \tag{13}$$

and higher-weight primitives (the odd zeta values $\zeta^{\mathfrak{m}}(2k+1)$, etc.) that no iterated cobracket of logarithms can reach. The gap Part IV closes is the *absence of an operation*: Parts I–III supply a domain, functions, and an object, but not the arrow that *breaks the object apart*. Part IV's emergent property is that arrow — the *decomposition law of observables*: an observable is not an atom but a word in a graded Lie coalgebra, spelled from irreducible primitives, and the cobracket is the grammar of its cuts. Two worked examples anchor it — the dilogarithm (Section 6.1) and the one-loop cut computed as the first coaction slot.

Remark 3.4 (The hand-off IV → V; status S). Part IV's Conditional Amplitude Decomposition transports the coaction through *any* Tannakian realisation homomorphism r_α. Part V selects the single realisation $\alpha = \mathrm{per}$ that produces honest numbers: every other r_α (Betti, de Rham, étale) rearranges structure; only the period/regulator realisation measures. The numerical period map is not a Hopf homomorphism (it lands in $\mathbb{C}$, which carries no weight grading) but a *comodule coaction* $\rho : \mathcal{P} \to \mathcal{H} \otimes \mathcal{P}$ whose first slot computes the discontinuity.

3.5 Part V: passage from motive to measurable number

Part V supplies the arrow the ladder was missing: the *period pairing* $\mathrm{per}(\Pi) = \int_\Gamma \omega$ for a period datum $\Pi = (X, D, \omega, \Gamma)$, and the *regulator* $r_\alpha : K_\bullet(X) \otimes \mathbb{Q} \to H^\bullet_{\mathcal{D}}(X, \mathbb{Q}(n))$ from algebraic K-theory (motivic cohomology) to Deligne–Beilinson cohomology. Its rigorous backbone is the multiplicativity/additivity of per (making per a ring homomorphism compatible with amplitude factorisation, so (2) interacts correctly with cuts), and its arithmetic depth is Borel's rank-and-covolume theorem tying the regulator lattice to Dedekind zeta values, generalised by Beilinson's conjecture. Its showcase is the *Bloch–Wigner regulator*

$$r : \; K_3^{\mathrm{ind}}(F) \otimes \mathbb{Q} \;\xrightarrow{\sim}\; B(F) \otimes \mathbb{Q} \;\xrightarrow{D}\; \mathbb{R}, \tag{14}$$

where $B(F) = \ker\bigl(\delta_2 : B_2(F) \to \bigwedge^2 F^\times\bigr)$ is the *Bloch group*, the kernel of the cobracket on the *pre-Bloch group* $B_2(F) = \mathbb{Z}[F \setminus \{0,1\}]/(\text{five-term})$; by Suslin's theorem $K_3^{\mathrm{ind}}(F) \otimes \mathbb{Q} \cong B(F) \otimes \mathbb{Q}$. The regulator's very well-definedness is that D satisfies the five-term relation: D_* then descends from the free group to the pre-Bloch group $B_2(F)$, and its restriction to $B(F) \subseteq B_2(F)$ is the regulator. (The parallel algebraic fact — the vanishing, in $\bigwedge^2 F^\times$, of the cobracket of the five-term combination $\sum(-1)^i\{\,[\widehat{z}_i]\,\}$ — is what lets δ itself descend to $B_2(F)$.) The gap Part V closes is the *absence of a number*: Part IV anatomises an observable but produces no measurement; Part V evaluates. Its emergent property is the passage from motive to measurable real number — and, crucially, that real number re-enters Part I's domain, which is the subject of Section 5.

4 A Unified Mathematical Framework: The Five Faces of One Object

The recomposition of Section 3 is a *chain* of hand-offs. We now give the *static* unification: a single object of which the five rungs are five faces. This is the synthesis's mathematical core.

4.1 The central object

Definition 4.1 (The library object; status S)**.** Let F be a number field (or a field of kinematic invariants of characteristic 0). The *library object* is the triple

$$\Big(\mathcal{H}(F),\ \mathcal{L}_G(F) = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2,\ \mathrm{per} : \mathcal{H}(F) \to \mathbb{C}\Big), \tag{15}$$

where $\mathcal{H}(F) = \bigoplus_{n\geq 0} \mathcal{H}_n(F)$ is the connected graded commutative Hopf algebra of framed mixed Tate motives over F (equivalently, of motivic iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and its cyclotomic variants), $\mathcal{L}_G(F)$ its Lie coalgebra of indecomposables with cobracket δ of (12), and per the period realisation of (11). The object is graded by weight and connected ($\mathcal{H}_0 = \mathbb{Q} \cdot 1$).

The library object exists unconditionally over a number field (Part III; Bloch–Kriz, Deligne–Goncharov, Levine, Brown), so nothing below is vacuous. We now read each rung off it.

Theorem 4.2 (Five Faces of one object; status S as mathematics, H in physical reading)**.** *Let* $\big(\mathcal{H}(F), \mathcal{L}_G(F), \mathrm{per}\big)$ *be the library object of Theorem 4.1. Then the five rungs of the ladder are five faces of this single object:*

(I) Domain. *The weight-one part is Part I's domain:* $\mathcal{H}_1(F) = \mathcal{L}_G(F)_1 = P_1 \cong F^\times \otimes \mathbb{Q}$*, the ratios of Part I, with period* $\log x$*.*

(II) Functions. *The distinguished elements* $\mathrm{Li}^{\mathfrak{m}}_{n_1,\ldots,n_d}(x_1, \ldots, x_d) \in \mathcal{H}_n(F)$*,* $n = \sum n_i$*, are the motivic lifts of Part II's polylogarithms; the weight grading of (5) is the internal grading of* $\mathcal{H}(F)$*, and the symbol is the maximal iteration of* Δ' *into* $\mathcal{H}_1^{\otimes n}$*.*

(III) Object. $\mathcal{H}(F)$ is *the coordinate ring of the pro-unipotent radical of the Tannakian fundamental group of* $\mathrm{MTM}(F)$*; a motivic amplitude object* $\mathcal{A}^{\mathfrak{m}}$ *is an element of* $\mathcal{H}(F)$*, and its four realisations are the four fibre functors — Part III's motive made into an algebra of functions on its own motivic Galois group.*

(IV) Decomposition. $\mathcal{L}_G(F) = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2$ *with the cobracket* δ *of (12) is Part IV's Goncharov Lie coalgebra; the graded dual* $\mathcal{L}_G(F)^\vee$ *is the free graded Lie algebra of the motivic Galois group, cogenerated by the primitives* P*.*

(V) Measurement. $\mathrm{per} : \mathcal{H}(F) \to \mathbb{C}$ *is Part V's period pairing; its induced comodule coaction* $\rho : \mathcal{P} \to \mathcal{H}(F) \otimes \mathcal{P}$ *is Brown's coaction, whose first slot is the discontinuity. The period map* per *itself, restricted to the weight-two primitives* $P_2 = \ker \delta \cong K_3^{\mathrm{ind}}(F) \otimes \mathbb{Q}$*, is the Bloch–Wigner regulator (14).*

Weight	$\mathcal{H}(F)$ / $\mathcal{L}_G(F)$	Part II function	Part V measurement
0	$\mathcal{H}_0 = \mathbb{Q}\cdot 1$	constants	rational numbers
1	$\mathcal{L}_G(F)_1 \cong F^\times \otimes \mathbb{Q}$ (all primitive)	$\log x$	per $=$ $\log x$ (Kummer; $\log\lvert x\rvert \in \mathbb{R}$ up to branch)
2	$\mathcal{L}_G(F)_2 \cong B_2(F) \otimes \mathbb{Q}$ (pre-Bloch); $P_2 = \ker\delta \cong B(F)\otimes\mathbb{Q}$	$\mathrm{Li}_2(x)$	Bloch–Wigner D on $K_3^{\mathrm{ind}}(F) \cong B(F)$
3	$\mathcal{L}_G(F)_3$; $P_3 \ni \zeta^{\mathfrak{m}}(3)$	$\mathrm{Li}_3, \mathrm{Li}_{2,1}$	$K_5^{\mathrm{ind}}(F)$ regulator (higher)
n	$\mathcal{L}_G(F)_n$; $\delta : \mathcal{L}_G(F)_n \to \bigoplus_{p+q=n} \mathcal{L}_G(F)_p \wedge \mathcal{L}_G(F)_q$	weight-n MPLs	Beilinson regulator on $K_{2n-1}(F)$

Table 4: The weight grading is the common spine of all five rungs. Reading a column downward is a single rung; reading a row across is the synthesis: at each weight the same integer is a transcendentality (II), a coalgebra grading (IV), and the argument of a regulator (V).

Proof. Each clause is a citation to the corresponding rung, assembled here. (I) is the weight-one Kummer computation of Parts I and III, $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q}$, together with Part IV's identity $P_1 = \mathcal{L}_G(F)_1$ (weight one cannot split as $p+q$ with $p, q \geq 1$, so $\delta|_{\mathcal{L}_G(F)_1} = 0$). (II) is Part III's identification of multiple polylogarithms as periods of $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ and Part IV's symbol as maximal iteration of Δ' (Sections 3.2 and 3.4). (III) is the Tannakian formalism of Part III: $\mathrm{MTM}(F)$ is neutral Tannakian, its fundamental group $G_{\mathrm{MTM}(F)}$ is an extension $1 \to U \to G \to \mathbb{G}_m \to 1$ with U pro-unipotent, and $\mathcal{H}(F)$ is the graded coordinate ring of U. (IV) is Part IV's well-definedness and cogeneration theorems, with freeness of $\mathcal{L}_G(F)^\vee$ following from Borel's computation of $\dim_{\mathbb{Q}} \mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(n))$. (V) is Part V's period map together with Part IV's comodule remark identifying ρ with Brown's coaction. The physical reading of each clause ("domain of observables," "loop order," "pre-numerical object," "decomposition law," "measurement") is the H overlay of the respective rung; by Theorem 2.1 the conjunction is H as physics and S as mathematics. $\square$

Theorem 4.2 is the precise sense in which the library is not five separate theories but five readings of one object. The modular presentation is pedagogical and epistemic — it lets each rung be judged on its own — but the underlying mathematics is unified: $\mathcal{H}(F)$ *carries the domain in weight one, the functions as elements, the object as its own coordinate ring, the decomposition as its indecomposables, and the measurement as its period map.*

4.2 The graded anatomy in one display

Table 4 tabulates the five faces against the weight grading, making visible that the grading is the common spine.

5 The Closure Theorem: Measurement as the Closure of the Ladder

We now prove the capstone. The five rungs compose to the master formula observable = per(motive), and the output re-enters Part I's domain, so the ladder closes into a loop. Measurement is the arrow that closes it.

5.1 The composite pipeline

Definition 5.1 (The realisation pipeline; status S as mathematics, H as physics)**.** For a tame (mixed Tate) n-point kinematic configuration, the *realisation pipeline* is the composite of the five hand-offs of Section 3:

$$\underbrace{F^\times}_{\text{I}} \xrightarrow{\lambda} \underbrace{\mathrm{Li}_n(x)}_{\text{II}} \xrightarrow{\text{Mot}} \underbrace{\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}(F)}_{\text{III}} \xrightarrow{\Delta',\delta} \underbrace{\mathcal{L}_G(F)}_{\text{IV}} \xrightarrow{\text{per}} \underbrace{\mathcal{A} \in \mathcal{P}}_{\text{V}}, \tag{16}$$

where λ assigns to a cross-ratio the polylogarithm evaluated there, Mot realises the polylogarithm's value as a period of a motivic object, Δ', δ are the decomposition operations, and per evaluates. The composite Real := per $\circ$(lift) realises the master slogan (1).

The pipeline is not a single functor $\mathsf{Math} \to \mathsf{Phys}$; it is a composite of domain-specific arrows, each defensible on its own rung. This is the modular discipline made mathematical.

5.2 The closure statement

Theorem 5.2 (Closure Theorem; status S as mathematics, H as physical reading)**.** *Let F be a number field and let $\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}(F)_n$ be a tame (mixed Tate) motivic amplitude object of weight n, with numerical realisation $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$. Then:*

(1) (Decomposition, Part IV.) *The coalgebraic anatomy of $\mathcal{A}^{\mathfrak{m}}$ under iterated Δ' is a finite rooted tree of depth $\leq n$, whose leaves are irreducible primitives: the weight-one logarithms $\log x$, $x \in F^\times$ (that is, $P_1 \cong F^\times \otimes \mathbb{Q}$, Part I's ratios), together with the irreducible higher-weight constants of $\bigoplus_{k \geq 2} P_k$ (such as the odd zeta values), each with vanishing cobracket.*

(2) (Realisation, Parts III, V.) $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}}) = \int_\Gamma \omega$ *is the period pairing of a period datum* $\Pi = (X, D, \omega, \Gamma)$ *representing $\mathcal{A}^{\mathfrak{m}}$; and* per *is a ring homomorphism (multiplicative and additive), so the realisation is compatible with the factorisation predicted by the cobracket of* (1).

(3) (Compatibility, Part IV/V.) *The numerical realisation intertwines decomposition and measurement: the comodule coaction $\rho(\mathcal{A}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathrm{per}(\mathcal{A}^{\mathfrak{m}}_{i,2})$ holds, and its first slot computes the physical discontinuity,* $\mathrm{Disc}_s \mathcal{A} = 2\pi i \sum_{i:\mathcal{A}^{\mathfrak{m}}_{i,2} = \log^{\mathfrak{m}} s} \mathrm{per}(\mathcal{A}^{\mathfrak{m}}_{i,1})$.

(4) (Closure, Part V $\to$ Part I.) *The real (or complex) number $\mathcal{A} \in \mathcal{P}$ is itself an element of a Part I domain: it generates a field $F' := F(\mathcal{A})$ whose multiplicative group $F'^\times$ is a new disciplined domain of observable ratios, into which $\mathcal{A}$ may be fed as a measured scale of a fresh kinematic configuration.*

Consequently the composite (16) *reproduces observable* = per(*motive*) (2)*, and the extended pipeline* I → II → III → IV → V → I *closes into a loop. The arrow* V → I *that closes it is precisely* measurement*.*

Proof. Each clause is a component theorem of the ladder, assembled under Theorem 2.1.

(1) is Part IV's termination and cogeneration theorems. By Section 3.4, the reduced coproduct on a connected graded Hopf algebra strictly lowers weight in each tensor factor (both factors of $\Delta'(x)$ have weight ≥ 1 summing to n, hence each $\leq n-1$); so iterating Δ' on a weight-n element terminates after at most n steps, and each branch stops at a *primitive* (an element with $\delta = 0$). The weight-one primitives are $P_1 \cong F^\times \otimes \mathbb{Q}$ by (13); higher-weight primitives exist (e.g. $\zeta^{\mathfrak{m}}(3) \in P_3$) and are irreducible informational units no iterated cobracket of logarithms reaches. Finiteness follows from finite-dimensionality of each weight-graded piece and depth bound n.

(2) is Part III's period interpretation (11) together with Part V's multiplicativity theorem: per is $\mathbb{Q}$-bilinear in (ω, Γ) and multiplicative on the product period datum $\Pi_1 \otimes \Pi_2$ (Fubini for products of chains, i.e. the Künneth decomposition at the level of the comparison isomorphism), so the period ring $\mathcal{P}$ is closed under $+$ and $\times$ and per is a ring homomorphism. Hence a factorised amplitude $\mathcal{A} = \mathcal{A}_1\mathcal{A}_2$ with $\mathcal{A}^{\mathfrak{m}} = \mathcal{A}_1^{\mathfrak{m}} \otimes \mathcal{A}_2^{\mathfrak{m}}$ realises as $\mathrm{per}(\mathcal{A}_1^{\mathfrak{m}} \otimes \mathcal{A}_2^{\mathfrak{m}}) = \mathrm{per}(\mathcal{A}_1^{\mathfrak{m}})\,\mathrm{per}(\mathcal{A}_2^{\mathfrak{m}}) = \mathcal{A}_1\mathcal{A}_2$.

(3) is Part IV's comodule identity: the period map is not a Hopf homomorphism but the $\mathcal{H}$-comodule coaction $\rho : \mathcal{P} \to \mathcal{H} \otimes \mathcal{P}$ (Brown's coaction), which intertwines analytic continuation with the coproduct. The monodromy around $s = 0$ acts on the Betti realisation by Picard–Lefschetz, whose logarithm is the weight-one nilpotent attached to $\log^{\mathfrak{m}} s$; dually this picks out exactly the coaction terms whose second slot is $\log^{\mathfrak{m}} s$, with coefficient $2\pi i = \mathrm{per}(\mathbb{Q}(1))$, giving the discontinuity formula.

(4) is elementary and is the closure. $\mathcal{A} \in \mathbb{C}$ is a period; if $\mathcal{A}$ is real (as for the Bloch–Wigner regulator, a Borel covolume, or a log of a positive ratio) it lies in $\mathbb{R}$, and in all cases $F' := F(\mathcal{A})$ is a field extension of F inside $\mathbb{C}$. By Part I's Definition of a kinematic domain, any such field is a legitimate domain of observable values, and its multiplicative group $F'^\times$ houses $\mathcal{A}$ as a nonzero scale/ratio (assuming $\mathcal{A} \neq 0$; the degenerate case $\mathcal{A} = 0$ is the vanishing of an observable, itself a datum). Thus the output of Part V is an input to a fresh instance of Part I.

Assembling (1)–(4): the composite (16) sends a Part I ratio through functions (II), object (III), decomposition (IV), and evaluation (V) to a number $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$, which is (2); and by (4) that number re-enters Part I. The mathematics of each step is S; the reading of the composite as "physical measurement of an observable" is H, and by Theorem 2.1 the composite is H as physics and S as mathematics. The tame hypothesis (that $\mathcal{A}^{\mathfrak{m}}$ lies in the mixed Tate $\mathcal{H}(F)$) is essential; Section 8 records where it fails. □

Remark 5.3 (What "closure" does and does not claim; status H). The closure is not the trivial observation that "a measured number is a number." It is the structural statement that the *same* algebraic object — $F^\times \otimes \mathbb{Q}$ in weight one — sits at both ends of the pipeline: it is Part I's domain (Section 3.1) *and* Part IV's weight-one primitives ((13)) *and* the image of the weight-one period map $\log : F^\times \to \mathbb{R}$ (Table 4). Measurement does not leave the category of the library; it returns to the category's ground floor. The loop is a loop *in the same object*, not a vague analogy between input and output. This is why Theorem 5.2(4) is a genuine synthesis statement rather than a tautology: it is the composition of Theorem 4.2(I) with Theorem 4.2(V) closing on the identical group $F^\times \otimes \mathbb{Q}$.

5.3 Measurement as the evaluation of surviving primitives

The closure has a sharp reading through the cobracket–regulator square of Part V. Display, for weight two,

$$\begin{tikzcd} K_3^{\mathrm{ind}}(F)\otimes\mathbb{Q} \cong P_2 = \ker\delta \arrow[r, hook] \arrow[d, "r = D_*"'] & \mathcal{L}_G(F)_2 \arrow[d, "\delta"] \\ \mathbb{R} & \Lambda^2(F^\times\otimes\mathbb{Q}) \arrow[l, dashed, "\text{(no map)}"'] \end{tikzcd} \tag{17}$$

The right arrow is Part IV's weight-two cobracket $\delta\{x\} = (1-x)\wedge x$ on the whole of $\mathcal{L}_G(F)_2 \cong B_2(F)\otimes\mathbb{Q}$; the top arrow is the inclusion of the primitives $P_2 = \ker\delta \cong B(F)\otimes\mathbb{Q} \cong K_3^{\mathrm{ind}}(F)\otimes\mathbb{Q}$; the left arrow is Part V's regulator $r = D_*$. The bottom is dashed because *no* map completes the square: the regulator is defined precisely on the *kernel* of the cobracket (the Bloch group $B(F) = \ker\delta$), the classes whose cobracket vanishes, whereas δ itself is nontrivial on the complementary part of $\mathcal{L}_G(F)_2$. This is the exact content of "measurement is the evaluation of the primitives that survive the decomposition law": the cobracket detects the information split of an observable, and the regulator measures precisely the part that does not split further.

Corollary 5.4 (Measurement measures primitives; status S, physical reading H)**.** *In weight two, the Bloch–Wigner regulator (14) is well-defined on the Bloch group $B(F) = P_2 = \ker\big(\delta : \mathcal{L}_G(F)_2 \to \bigwedge^2 F^\times\big)$ if and only if D respects the five-term relation. This condition is the analytic twin of the vanishing, in $\bigwedge^2 F^\times$, of the cobracket of the five-term combination $\sum(-1)^i\{[\widehat{z}_i]\}$: the same relation makes δ descend to $\mathcal{L}_G(F)_2$ and makes D_* descend to it. Measurement is then the evaluation of D_* on the primitives $P_2 = \ker\delta$ — the weight-two classes that are cut-free.*

Proof. By Part II's five-term relation (18), the extended-linearly function $D_* : \mathbb{Z}[F\backslash\{0,1\}] \to \mathbb{R}$, $\sum n_j[x_j] \mapsto \sum n_j D(x_j)$, annihilates the five-term relations, so it descends to the pre-Bloch group $B_2(F) = \mathcal{L}_G(F)_2$ (this descent *is* the well-definedness). Restricting the descended D_* to the Bloch group $B(F) = \ker(\delta : \mathcal{L}_G(F)_2 \to \bigwedge^2 F^\times) = P_2$ and composing with Suslin's isomorphism $K_3^{\mathrm{ind}}(F)\otimes\mathbb{Q} \cong B(F)\otimes\mathbb{Q}$ gives the regulator on K-theory. The five-term combination's cobracket vanishes in $\bigwedge^2 F^\times$ — which is what allows δ to descend to $\mathcal{L}_G(F)_2$ in the first place — so the whole square (17) is consistent. □

6 Emergent Properties from Composition

The synthesis's payoff is that composition creates properties no single rung possesses. We give three, in increasing order of scope, then the capstone.

6.1 The golden thread: the five-term relation read five ways

The single richest worked example in the library is the five-term relation for the dilogarithm. Fix the Rogers dilogarithm $L(x) = \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x)$ and the single-valued Bloch–Wigner function $D(z) = \mathrm{Im}\,\mathrm{Li}_2(z) + \arg(1-z)\log|z|$. The relation

$$D(u) + D(v) + D\Big(\tfrac{1-u}{1-uv}\Big) + D(1-uv) + D\Big(\tfrac{1-v}{1-uv}\Big) = 0 \tag{18}$$

is one identity that is read differently on every rung.

Principle 6.1 (The five-term relation is a single object read five ways; status as marked). The relation (18) is simultaneously:

(I) *Kinematic (*S*).* Its arguments are the five cross-ratios of five points $z_0, \dots, z_4 \in \mathbb{P}^1$; in cross-ratio form $\sum_{i=0}^{4}(-1)^i D([\widehat{z_i}]) = 0$ it lives on $\mathrm{Conf}_5(\mathbb{P}^1)$ and is invariant under the anharmonic group. It is a statement about Part I's domain.

(II) *Functional (*S*).* It is the fundamental functional equation of the weight-two polylogarithm — the relation that all one-variable dilogarithm identities reduce to (Wojtkowiak). It is a statement about Part II's functions.

(III) *Motivic (*S*).* It is a relation among the periods of the weight-two framed mixed Tate motives $\mathrm{Li}_2^{\mathfrak{m}}(x)$; the five terms are five realisations of subquotients of one motivic configuration. It is a statement about Part III's object.

(IV) *Coalgebraic (*S*, physical reading* H*).* It is the defining relation of the *pre-Bloch group* $B_2(F) = \mathbb{Z}[F \setminus \{0,1\}]/(\text{five-term})$, which satisfies $B_2(F) \otimes \mathbb{Q} \cong \mathcal{L}_G(F)_2$: the five-term combination in the free group $\mathbb{Z}[F \setminus \{0,1\}]$ has *vanishing cobracket* in $\bigwedge^2 F^\times$ ($\delta\{x\} = (1-x) \wedge x$ and the signed sum cancels), which is exactly what lets δ descend to the quotient $\mathcal{L}_G(F)_2$. It is a statement about Part IV's decomposition law.

(V) *Regulator-theoretic (*S*).* It is exactly the well-definedness condition of the Bloch–Wigner regulator (14) on $K_3^{\mathrm{ind}}(F) \cong B(F)$: because D vanishes on the five-term combination, the map D_* descends to $B_2(F) = \mathcal{L}_G(F)_2$, and its restriction to the Bloch group $B(F) = \ker\delta = P_2$ is the regulator. (Distinctly, the cobracket δ descends to $B_2(F)$ because the same combination's *cobracket* vanishes in $\bigwedge^2 F^\times$.) It is a statement about Part V's measurement.

No single rung sees more than its own reading; the emergent fact — that these are one identity — is visible only in the synthesis.

Proof. (I) is the PGL_2-normalisation of five points and the anharmonic invariance of $D \circ r$ (Part I, Part II). (II) is the equivalence of the Rogers, Bloch–Wigner, and cross-ratio forms by substitution of five-point cross-ratios (Part II). (III) is the motivic lift $\mathrm{Li}_2^{\mathfrak{m}}$ of Part III. (IV) is Part IV's computation $\delta\,\mathrm{Li}_2^{\mathfrak{m}}(x) = \log^{\mathfrak{m}}(1-x) \wedge \log^{\mathfrak{m}}(x) = (1-x) \wedge x$ and the verification, by bilinearity and antisymmetry of $\wedge$, that the signed five-term sum has vanishing cobracket in $\bigwedge^2 F^\times$; Zagier–Bloch–Suslin show this is exactly the relation defining $B_2(F) \otimes \mathbb{Q} \cong \mathcal{L}_G(F)_2$ and that $\ker\delta \cong B(F) \otimes \mathbb{Q}$. (V) is Part V's regulator theorem (Theorem 5.4). The statuses are those the individual parts assign. □

This is the tightest illustration of the library's thesis: *the rungs are not analogies stacked side by side; they are readings of one object.* The five-term relation is the coalgebra's grammar (IV) made visible simultaneously as geometry (I), analysis (II), arithmetic (III), and measurement (V).

6.2 One integer read three ways: the weight grading

A second emergent property concerns the weight grading. On Part II it is transcendental complexity (and, heuristically, loop order); on Part III it is (up to sign and doubling) the motivic/Hodge weight of the framed mixed Tate motive; on Part IV it is the internal grading of $\mathcal{L}_G(F)$ that the cobracket strictly lowers. These are *a priori* three different integers attached to a weight-n object; the emergent fact is that they coincide.

Proposition 6.2 (Weight coherence; status S, physical reading S/H)**.** *For a framed mixed Tate motive underlying a weight-n multiple polylogarithm, the following three gradings agree (after the standard normalisation identifying the motivic weight of $\mathbb{Q}(k)$, namely $-2k$, with the transcendental weight k): the transcendental weight of Part II, the (normalised) motivic weight of Part III, and the Lie-coalgebra grading of Part IV. Consequently the single slogan "weight = loop order" (S/H) and the single fact "δ lowers weight" (S) refer to one integer.*

Proof. The motivic dilogarithm $\mathrm{Li}_2^{\mathfrak{m}}(x)$ has period matrix a unipotent lower-triangular 3×3 matrix with diagonal $1, 2\pi i, (2\pi i)^2$ recording the Tate twists $\mathbb{Q}(0), \mathbb{Q}(1), \mathbb{Q}(2)$ of motivic weights $0, -2, -4$; the transcendental weights $0, 1, 2$ are the negatives-halved. In general $\mathrm{Li}^{\mathfrak{m}}_{n_1,\ldots,n_d}$ is an iterated extension involving $\mathbb{Q}(0), \ldots, \mathbb{Q}(n)$, so its transcendental weight $n = \sum n_i$ equals the top Tate twist index, which is the (normalised) motivic weight. The Lie-coalgebra grading is by definition the weight grading of $\mathcal{H}(F)$ restricted to indecomposables, and Δ' (hence δ) is graded of degree 0 while strictly lowering the weight of each tensor factor. All three gradings are therefore the same $\mathbb{Z}_{\geq 0}$-grading of $\mathcal{H}(F)$. The physical reading (that this integer is loop order) is the H overlay of Part II, established case-by-case, not by a universal theorem. $\square$

6.3 Cut and regulator: complementary operations on one coaction

A third emergent property, flagged in Parts IV and V as a novel and open direction, is the relationship between the *cut* (an operation that lowers weight by extracting a coaction slot) and the *regulator* (an operation that evaluates a class to a number). Part IV's discontinuity formula and Part V's regulator are two operations on the *same* comodule coaction ρ: the cut reads the first tensor slot (Theorem 5.2(3)), while the regulator evaluates the primitive that survives all cuts (Theorem 5.4). The synthesis observes that they are defined on complementary parts of one group — the cobracket on all of $\mathbb{Z}[F \setminus \{0, 1\}]$, the regulator on its kernel — and conjectures their precise relationship.

Remark 6.3 (Cut/regulator complementarity; status H). On the weight-two nose, (17) shows the cobracket δ and the regulator r act on complementary parts of one group: δ is nonzero on classes that split, r is defined on classes that do not. Heuristically, "taking a discontinuity" lowers weight one step (peels a coaction slot) while "measuring" evaluates the cut-free residue; a precise general statement — that discontinuity and regulator are adjoint operations on the motivic coaction — is, to this library's knowledge, open (Section 9). We flag it as a genuinely emergent, physically motivated problem that only the composed ladder makes visible.

6.4 The capstone: measurement as an emergent property

The three properties above culminate in the capstone. Each individual rung lacks the concept of *measurement*: Part I has a domain but no functions to measure; Part II has functions but no notion that their values are periods; Part III has periods but no decomposition; Part IV has a decomposition but produces no number; Part V produces numbers but, in isolation, is a chapter of arithmetic geometry with no observable to measure. Measurement — the assertion that a physical observable *is* the period pairing of the motivic decomposition of a kinematic configuration — exists only in the composite.

Principle 6.4 (Measurement is the emergent capstone; status S (spine), H (reading)). Let the library object be as in Theorem 4.1. Then *measurement*, understood as the map sending a kinematic configuration to the real number an experiment reports, is not a property of any single rung but the emergent property of the closed composite $\mathrm{I} \to \mathrm{II} \to \mathrm{III} \to \mathrm{IV} \to \mathrm{V} \to \mathrm{I}$ of Theorem 5.2. It is the closure of the loop: the regulator/period pairing of Part V (Theorem 4.2(V)) closes the loop opened by the domain of observables of Part I (Theorem 4.2(I)) on the identical object $F^\times \otimes \mathbb{Q}$. Symbolically,

$$\text{measurement} \;=\; \big[\,\mathrm{V} \to \mathrm{I}\,\big] \;=\; \big(\mathrm{per} : \mathcal{H}(F) \to \mathbb{C}\big)\big|_{\text{amplitude } \mathcal{A}^{\mathfrak{m}}} \;\longrightarrow\; \mathbb{R} \hookrightarrow F'^{\times}. \tag{19}$$

Here per evaluates the *whole* motivic amplitude object $\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}(F)$ (decomposable products included); the regulator of Theorem 5.4 is the restriction of this evaluation that isolates the *surviving primitives* $P = \ker \delta$, the cut-free constituents whose measurement carries the irreducible arithmetic content.

Proof. By Theorem 5.2 the composite reproduces $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ and re-enters $F^\times$; by Theorem 5.4 the evaluation is defined on the primitives surviving the decomposition; and by Theorem 4.2 the two ends of the loop are the same weight-one object $F^\times \otimes \mathbb{Q}$. No sub-composite omitting a rung yields a map from a kinematic configuration to a measured number: dropping I removes the domain and the closure target; dropping II removes the functions realised by per; dropping III removes the object $\mathcal{A}^{\mathfrak{m}}$ that per acts on; dropping IV removes the decomposition whose surviving primitives are measured; dropping V removes per itself. Hence measurement is emergent in the composite and in no proper sub-composite. The spine is S; the identification of the composite with *physical* measurement is H. □

7 The Epistemic Audit

Following Theorem 2.1, we tabulate the status labels of the library's principal claims. The audit is a quantitative honesty device: it shows at a glance that the structural spine is S, the physical readings are quarantined as H, and only the cosmic Galois group and the general Beilinson conjecture reach $\mathsf{H/P/H}$.

$$\begin{tikzcd}
& \mathcal{H}(F)\ \text{(Part III object)} \arrow[dl, "\Delta'\ \text{(IV)}"'] \arrow[dr, "\mathrm{per}\ \text{(V)}"] & \\
\mathcal{L}_G(F),\ \delta\ \text{(Part IV)} \arrow[dr, "\text{primitives } P"'] & & \mathcal{A} = \int_\Gamma \omega\ \text{(Part V)} \arrow[dl, "\in\mathbb{R}\hookrightarrow F'^{\times}"] \\
& F^\times \otimes \mathbb{Q}\ \text{(Part I domain} = P_1\text{)} &
\end{tikzcd}$$

Figure 2: The closure triangle. The Part III object decomposes (IV) into primitives whose weight-one part is Part I's domain $F^\times \otimes \mathbb{Q} = P_1$, and realises (V) to a number that re-enters a Part I domain $F'^\times$. Measurement is the descent of both legs onto the ground floor $F^\times \otimes \mathbb{Q}$.

Remark 7.1 (Reading the audit; status S). Of the roughly two dozen principal claims (Table 5), the large majority are S as mathematics. Every physical reading that names a rung's emergent property ("domain of observables," "weight = loop order," "decomposition law," "measurement") is S/H or H, never silently upgraded. Exactly one claim is H/P — the cosmic Galois group of Part IV — and it is never composed into a load-bearing position. This distribution is the quantitative form of the library's central discipline: *the mathematics is standard; the physics is an honestly flagged interpretive overlay.*

8 Limitations

We reproduce and address the library's limitations under the same discipline as each part.

1. **Not every analogy is a theory.** The strongest results of every rung — Theorems 4.2 and 5.2 included — are S as mathematics; their universal *physical* reading (that any observable is the period of a motive, that measurement is the composite) is H. By Theorem 2.1 the physical composite inherits H. The S/H/P labels are part of the formalism.

2. **Motives are not "the functions of" an object.** The functional-essence reading of Part III (motive as "functional essence") is a proposed semantic layer, status H, and is never presented as Grothendieck's definition. A motive is the universal cohomological invariant, not the ring of functions $\mathcal{O}(X)$.

3. **No universal $\mathsf{Math} \to \mathsf{Phys}$ functor.** The pipeline (16) is a composite of local, domain-indexed arrows, not a single global functor. The modularity is not a stylistic choice but a mathematical necessity: each realisation channel is local.

4. **The Non-Tate boundary.** The tame, weight-graded polylogarithmic anatomy of Theorem 5.2 is not universal. When a motivic amplitude object's lift has, as a subquotient, the motive $H^1(E)$ of an elliptic curve E/F with $\mathrm{End}(E) = \mathbb{Z}$ (or another non-Tate simple motive), the comodule structure over the mixed Tate Hopf algebra fails: the period matrix contains the elliptic periods ω_1, ω_2 and quasi-periods η_1, η_2 (Legendre relation $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$), not $\mathbb{Q}$-combinations of multiple zeta values. The Closure Theorem then holds only relative to the enlarged elliptic-polylogarithm coalgebra (Levin–Racinet,

Broedel–Duhr–Dulat–Tancredi). This is a precise, citable limitation, not a vague caveat: the named obstruction is $H^1(E)$ as subquotient, the named remedy is the elliptic coalgebra, and it is where the plain weight count of Theorem 6.2 must be refined.

5. **Open conjectures at the top.** The cosmic Galois group (Part IV, H/P) and the general Beilinson conjecture (Part V, H) are the two most speculative/open components. They retain their labels in the synthesis and are not upgraded. In particular the closure does not depend on either: Theorem 5.2 uses only the unconditional $\mathrm{MTM}(F)$, the multiplicativity of per, and Borel's theorem (via its weight-two Bloch–Wigner case), all S.

6. **The residue-tree/coaction-tree identification is conjectural.** The most attractive structural bridge the library surfaces — that the positive-geometry boundary/residue tree of Part I equals the motivic coaction tree of Part IV — is open (Section 9). The synthesis neither assumes nor proves it.

9 Open Problems

The composed ladder makes several problems visible that no single rung poses. We collect them as the library's forward-looking research program.

1. **Residue tree = coaction tree.** Prove or disprove that the boundary/residue tree of a positive geometry (Part I: the Deligne–Mumford boundary of $\overline{\mathcal{M}}_{0,n}$, cluster-coordinate combinatorics) is isomorphic to the iterated-coaction tree of Part IV. This is arguably the single most interesting open problem the library raises; the five-term relation is the smallest case where both sides are fully explicit (the pentagon $\overline{\mathcal{M}}_{0,5}$ versus $\ker\delta$ in $\mathcal{L}_G(F)_2$).

2. **Cut/regulator adjunction.** Make precise the complementarity of Theorem 6.3: is "take a discontinuity" (Part IV) adjoint, in a suitable sense, to "apply a regulator" (Part V) on the motivic coaction? A theorem here would unify the amplitudes community's cut/discontinuity calculus with the arithmetic geometers' regulator machinery.

3. **Beilinson beyond the tame regime.** Extend the regulator/Beilinson formalism to the elliptic and modular non-Tate regime obstructed in Section 8: what is the correct "regulator" on the elliptic-polylogarithm coalgebra, and does an elliptic analogue of the Bloch–Wigner five-term consistency (Theorem 5.4) hold?

4. **Higher-weight measurement maps.** Using the recent K-theoretic reconstructions of the Goncharov Lie coalgebra (Kupers–Rudenko–Sierra; Greenberg–Kaufman–Li–Zickert), determine the weight-three and higher analogues of the Bloch–Wigner regulator — the measurement maps out of $K_5^{(3)}(F)$ and beyond — and their five-term-type consistency conditions.

5. **The cosmic Galois group for a general QFT.** Give a precise definition of the cosmic Galois group G_c acting on the periods of an arbitrary quantum field theory (Brown's

conjecture), and determine whether the recent reformulations of $\mathcal{L}_G(F)$ bear on it. This is the H/P apex of the ladder.

10 Formalisation

In the spirit of the library's commitment to formal transparency, the synthesis is accompanied by machine-checkable artifacts that exhibit the *composite* of the ladder.

Haskell. The module `Ladder` (in the topic's `src/` directory) provides a weight-graded model of the five rungs and their composition: a `Rung` enumeration and a `Weight`-indexed representation; the weight-one identification $P_1 \cong F^\times \otimes \mathbb{Q}$ realised as the Kummer period log; the weight-two cobracket $\{x\} \mapsto (1-x) \wedge x$ into $\Lambda^2 F^\times$; a numerical Bloch–Wigner function with a check of the five-term relation (18) and of $D(i) = G$ (Catalan) and $D(e^{i\pi/3}) = 1.01494\ldots$ (the regular ideal tetrahedron volume); and a `closure` routine witnessing Theorem 5.2(4) — that the period output of a weight-one class re-enters the domain as a new ratio. A `main` routine prints a labelled PASS/FAIL verification report and compiles under GHC with no dependency beyond `base`.

Lean 4. The file `Synthesis.lean` (in the topic's `lean/` directory) is self-contained (it does not import Mathlib) and declares, inside `namespace Synthesis`: the `Rung` inductive type; a `ladderStep` successor function and the composite `pipeline`; the weight grading; the statement that the cobracket lowers weight; and the *closure* statement that the realisation of a weight-one primitive is a new domain element. Statements provable from the elementary scaffolding are proved; the deep theorems (cogeneration, Suslin's isomorphism, Borel's covolume) are recorded as `theorem` with `sorry` and a comment naming the corresponding synthesis result, in the honest spirit of the S/H/P discipline.

11 Conclusion

We set out to recompose a modular library of five standalone parts into a coherent whole, and to extract the emergent property that only the whole possesses. The recomposition (Section 3) showed each hand-off $\mathrm{I} \to \mathrm{II} \to \mathrm{III} \to \mathrm{IV} \to \mathrm{V}$ to be a concrete mathematical fact: Part I's cross-ratio domain $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is precisely where Part II's polylogarithms live; those functions' values are periods of Part III's motivic fundamental group; that motive's Hopf algebra $\mathcal{H}(F)$ is decomposed by Part IV's coproduct and cobracket into a Lie coalgebra whose weight-one primitives are Part I's ratios; and Part V's period pairing evaluates the surviving primitives to a real number. The unified framework (Section 4) collapsed the chain into a single object: $\mathcal{H}(F)$ carries the domain in weight one, the functions as elements, the object as its own coordinate ring, the decomposition as its indecomposables, and the measurement as its period map — the *Five Faces* of Theorem 4.2. The *Closure Theorem* (Theorem 5.2) proved that the composite reproduces observable = per(motive) and that its numerical output re-enters Part I's domain on the identical group $F^\times \otimes \mathbb{Q}$, so the ladder closes into a loop rather than terminating.

The capstone emergent property (Theorem 6.4) is **measurement as the closure of the ladder**. Measurement belongs to no single rung: Part I has a domain but nothing to measure, Part V has a regulator but no observable. It is the arrow V $\to$ I that closes the loop — the evaluation of the primitives that survive the decomposition law, landing back in the ground floor of ratios from which the whole apparatus was built. The five-term dilogarithm relation (Theorem 6.1) is the golden thread that makes this concrete: one identity that is simultaneously kinematic, functional, motivic, a cobracket kernel, and a regulator consistency condition — the coalgebra's grammar read five ways.

The library's governing thesis is thereby made precise across all six rungs: *a Goncharov-style Lie coalgebra is the bookkeeping system for how physical observables decompose into irreducible informational constituents*, and measurement is the closure of that bookkeeping — the moment the pre-numerical anatomy is paired against a cycle to yield a number, which is itself a new observable ratio. Physics observes numerical shadows of deeper motivic structures; the synthesis shows those shadows form a loop, and that the loop's closure is measurement itself.

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Part	Principal claim	Status
I	Möbius invariance of the cross-ratio	S
I	$\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$	S
I	$F^\times \otimes \mathbb{Q} \cong \mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1))$	S
I	Domain reading of F, $F^\times$	S/H
II	Chen path-independence	S
II	Termination of the reduced coproduct	S
II	Five-term relation (three forms)	S
II	Weight = loop order	S/H
III	Existence of mixed realisations (Huber)	S
III	Unconditional $\mathrm{MTM}(F)$ over number fields	S
III	Recurrence via shared motives	S/H
III	Motive as "functional essence"	H
IV	Cobracket well-definedness & co-Jacobi	S
IV	Cogeneration; $P_1 \cong F^\times \otimes \mathbb{Q}$	S
IV	Conditional Amplitude Decomposition	S
IV	Decomposition law of observables	S/H
IV	Cobracket = physical cut	H
IV	Cosmic Galois compatibility	H/P
V	Multiplicativity of the period pairing	S
V	Borel rank & covolume theorem	S
V	Bloch–Wigner regulator on $K_3^{\mathrm{ind}}(F)$	S
V	Beilinson conjecture (general)	H
V	Regulator = measurement (universal)	S/H
Synth.	Five Faces (Theorem 4.2)	S/ H
Synth.	Closure Theorem (Theorem 5.2)	S/ H
Synth.	Measurement as capstone (Theorem 6.4)	S/ H

Table 5: Epistemic audit of the library's principal claims. The mathematical spine is uniformly S; the physical readings are H; the two most speculative components (cosmic Galois group, general Beilinson conjecture) are H/P and H. No composite is advertised above the reliability of its weakest link (Theorem 2.1).