Regulators and the Period Pairing: From Motivic Object to Measurable Number

Part V of *Mathematics ↔ Physical Representation: A Modular Library*

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Abstract

This is Part V, the final constructive rung, of the modular library *Mathematics ↔ Physical Representation*, whose governing thesis is that a Goncharov-style Lie coalgebra is a *bookkeeping system* for how physical observables decompose into irreducible informational constituents, and that a physical number is the *realization* of a pre-numerical motivic object along the pipeline observable = per(motive). Rungs I–IV built, in turn, the disciplined domain of observable ratios ($F^\times$), the transcendental functions valued on it (Li_n, iterated integrals), the pre-numerical object behind those functions (the motive), and the coalgebraic decomposition law that anatomizes it (the coproduct Δ and cobracket δ on the Goncharov Lie coalgebra $\mathcal{L}_G(F)$). What has been missing is the arrow that actually *evaluates* the structure: the map that turns a motivic object into the single real number an experimentalist could compare to data. This paper supplies it. We formalize the *period datum* $\Pi = (X, D, \omega, \Gamma)$ and the *period pairing* $\mathrm{per}(\Pi) = \int_\Gamma \omega$, and we introduce the *regulator map* $r_\alpha \colon K_\bullet(X) \otimes \mathbb{Q} \to H^\bullet_{\mathcal{D}}(X, \mathbb{Q}(n))$ from algebraic K-theory to Deligne–Beilinson cohomology. We prove (T1) multiplicativity and additivity of the period pairing, licensing $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ as a ring homomorphism compatible with amplitude factorization; (T2) Borel's rank-and-covolume theorem tying the regulator lattice to Dedekind ζ-values; and we state (T3) Beilinson's conjecture as the general L-value template. The centrepiece is a fully worked derivation, carried the whole way from the dilogarithm five-term relation of Rung II, of the *Bloch–Wigner regulator* $r \colon K_3^{\mathrm{ind}}(F) \to \mathcal{B}(F) \otimes \mathbb{Q} \xrightarrow{D} \mathbb{R}$: we show its very well-definedness *is* the vanishing, in $\mathcal{L}_G(F)$, of the cobracket combination $\sum(-1)^i D([\cdots])$, and we exhibit the concrete number the machine outputs ($D(i)$ equals Catalan's constant; $D(e^{i\pi/3})$ is the volume of the regular ideal tetrahedron). Throughout we retain the S/H/P epistemic-status discipline of the series, name the emergent property added by this rung — the passage from motive to measurable real number — and show how the regulator's output re-enters Rung I's domain $F^\times$, closing the ladder into a loop. A compiling Haskell formalization and a Lean sketch of the period/regulator/Bloch-group types accompany the theory.

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1 Introduction

1.1 The modular ladder and where Part V sits

The library *Mathematics ↔ Physical Representation* formalizes a single thesis:

> A Goncharov-style Lie coalgebra is not merely a formal algebraic object; it is a *bookkeeping system* for how physical quantities break apart into irreducible informational constituents.

A physical observable — a scattering amplitude, a Feynman period, an entropy, a special value of an L-function — is treated as the *numerical realization* of a deeper, pre-numerical, structured object (a *motive*), and the question "what is this observable made of?" is answered not by further computation but by a *coalgebraic decomposition law*. The library is deliberately *modular*: rather than positing a single universal functor from all of mathematics to all of physics, it builds a ladder of domain-indexed rungs, each defensible on its own, each composing with its neighbour to add one emergent structural property that the level below cannot express.

Rung	New structure	Emergent property
I	F, $F^\times$, cross-ratios, $M_{0,n}$	disciplined domain of observable values
II	Li_n, iterated integrals, symbol	transcendental depth = loop order
III	motives $\mathrm{Mot}(X)$, realizations, comparison iso.	the pre-numerical object behind measurement
IV	Δ, δ, Goncharov Lie coalgebra $\mathcal{L}_G(F)$	the decomposition law of observables
V	**regulators, period pairing** $\mathrm{per}(\Pi) = \int_\Gamma \omega$	**passage from motive to measurable number**
Synth.	recomposition observable = per(motive)	measurement as closure of the ladder

Rungs I–IV assemble an elaborate pre-numerical machine but never turn its crank. Rung I supplies the raw ratios $F^\times$; Rung II supplies the special functions Li_n valued on Rung I's domain; Rung III recognizes the specific transcendental numbers of Rung II as *periods* of one geometric-arithmetic object; Rung IV endows that object with a coproduct Δ and cobracket δ that expose its primitive pieces. At the end of Rung IV we possess a fully anatomized *motivic amplitude object* $\mathcal{A}^{\mathfrak{m}}$ — but $\mathcal{A}^{\mathfrak{m}}$ is not a number. *This paper builds the arrow that makes it one.*

1.2 The emergent property added by Part V

The structure introduced by this rung is the pair

$$\boxed{\operatorname{per}(\Pi) = \int_\Gamma \omega, \qquad \Pi = (X, D, \omega, \Gamma), \qquad r_\alpha \colon K_\bullet(X) \otimes \mathbb{Q} \longrightarrow H^\bullet_{\mathcal{D}}(X, \mathbb{Q}(n)).} \tag{1}$$

The first is the *period pairing*: a differential form ω (de Rham data, the integrand) paired against an integration cycle Γ (Betti data, the domain of accumulation) produces a complex number. The second is the *regulator map*: a map from the algebraic K-theory (equivalently, motivic cohomology) of a variety to a realization cohomology carrying real-analytic data, whose composite with a period pairing extracts genuine real numbers — the numbers appearing in Borel's theorem and the Beilinson conjectures.

The emergent property is exactly what its name says: **the passage from a motivic object to a measurable real number**. Rung IV could decompose an amplitude into its primitive constituents but produced no number; Rung V evaluates the constituents against a concrete cycle and returns the number. Crucially, that number is *itself* an element of Rung I's domain $F^\times$ (a scale, a ratio, a measured invariant), so the ladder does not terminate at measurement — it *closes into a loop*, re-entering its own ground floor. This loop closure is the subject of the synthesis rung; here we prove the arrow that makes it possible.

1.3 The epistemic-status discipline

Following the whole series, we tag every dictionary entry and every claim with one of three epistemic-status labels, and we regard these labels as *part of the formalism*, not as decoration. They are the series' core honesty device: a chain of translations is only as reliable as its weakest link.

Label	Meaning	Canonical example
S	standard use in mathematics or mathematical physics	the Borel regulator; the Bloch–Wigner function on $\mathcal{B}(F)$
H	strong heuristic representation-dictionary entry	regulator as "measurement/numerical extraction"
P	speculative philosophical ontology	physical reality as the realization layer of motivic information

Composite labels S/H and H/P occur where an entry is standard as pure mathematics but heuristic or speculative in its *physical* reading. We adopt the following rule verbatim.

Rule of composition (status monotonicity). If two representation entries or theorems compose, the status of the composite is the minimum under the order $\mathsf{S} < \mathsf{H} < \mathsf{P}$. A P-labelled step cannot be upgraded to S merely by composing it with otherwise-rigorous results.

1.4 Summary of results

- Section 2 reproduces the master Math→Physics dictionary of the series verbatim, together with the regulator-specific refinement of it, each entry carrying its S/H/P label.

- Section 3 formalizes the period datum $\Pi = (X, D, \omega, \Gamma)$ and the period map, identifies periods as matrix entries of the comparison isomorphism, and proves **Theorem 3.4** (multiplicativity and additivity of the period pairing), with **Theorem 3.6** showing periods respect factorized amplitudes. This is the rigorous backbone licensing the master formula $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$.

- Section 4 defines Deligne–Beilinson cohomology and the general regulator map, proves **Theorem 4.4** (Borel's rank/covolume theorem relating the regulator lattice to Dedekind ζ-values), and states **Theorem 4.7** (Beilinson's conjecture) as the general L-value template.

- Section 5 is the payoff: the dilogarithm five-term relation of Rung II is carried all the way to the **Bloch–Wigner regulator** $r\colon K_3^{\mathrm{ind}}(F) \to \mathcal{B}(F) \otimes \mathbb{Q} \xrightarrow{D} \mathbb{R}$ (**Theorem 5.5**), whose well-definedness *is* the vanishing of a cobracket combination in $\mathcal{L}_G(F)$. We compute concrete numbers.

- Section 6 tabulates the physical interpretation; Section 7 collects limitations, open problems, the novel cut-versus-regulator comparison, and the loop closure toward synthesis; Section 8 describes the accompanying Haskell and Lean formalizations.

2 The Math → Physics Dictionary

Every paper in the series reproduces, verbatim, the seed dictionary as its opening translation table; each rung then refines it in its own direction. We reproduce the seed table first, then the regulator-specific refinement.

2.1 The seed dictionary (reproduced verbatim)

Mathematics	**Physical representation**
Field F	Kinematic/coordinate domain of possible values
Element of $F^{\times}$	Ratio, cross-ratio, scale, observable parameter
Polylogarithm $\mathrm{Li}_n(x)$	Iterated propagation / layered amplitude contribution
Period	Measurable physical number obtained from geometry
Motive	Pre-numerical informational object behind the measurement

Mathematics	Physical representation
Lie algebra	Infinitesimal symmetry / generator structure
Lie coalgebra	Decomposition law of observables
Cobracket δ	Factorization channel, cut, boundary, or information split
Weight grading	Complexity depth / loop order / transcendental depth
Primitive element	Irreducible informational unit
Coproduct	Full hierarchical decomposition
Regulator map	Passage from motivic object to physical real number
Hodge/de Rham realization	Analytic form seen by calculation
Betti realization	Topological/path/integration-cycle form
Period pairing	Physical observable as pairing of form and cycle

The central slogans that recur across the series:

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observable = realization(motivic information)
coalgebra = grammar of physical decomposition
Physics observes numerical shadows of deeper motivic structures.
Physical reality is the realization layer of structured mathematical info.
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The two dictionary entries that this paper makes precise are highlighted: *Regulator map* $\leftrightarrow$ *passage from motivic object to physical real number*, and *Period pairing* $\leftrightarrow$ *physical observable as pairing of form and cycle*. Everything below is an elaboration of these two lines.

2.2 The regulator-specific refinement

Mathematics	Phys. representation	Semantic meaning	Status
Period pairing $\int_\Gamma \omega$	Observable amplitude contribution	Pairing of form and integration cycle	S
Differential form ω	Integrand / local observable density	What is being accumulated	S
Cycle Γ	Integration domain, path, history class	Where physical accumulation occurs	S
Regulator map r_α	Motive to analytic number	Measurement / numerical extraction	S/H
Period ring $\mathcal{P}$	Physically reachable transcendental constants	Numbers a calculation could produce	S/H

Mathematics	Phys. representation	Semantic meaning	Status
Beilinson conjecture	Special L-value $\leftrightarrow$ regulator determinant	Arithmetic $\leftrightarrow$ analytic bridge	H (gen.) / S (proven)
Borel regulator	Archimedean measurement of a K-class	Lattice covolume becoming a real number	S
Bloch–Wigner regulator	Weight-2 period pairing via $D(z)$	Worked passage motive $\to$ number	S

3 The Period Pairing

3.1 Master formulas and notation

We fix the shared notation of the series. F is a field, typically a number field or a field of kinematic invariants; $F^\times$ its multiplicative group. A variety X is smooth over $F \subset \mathbb{C}$; $D \subset X$ is a (reduced) normal-crossings divisor; ω is an algebraic differential form on $X \setminus D$; Γ is a relative homology cycle of complementary dimension. $\mathcal{A}^{\mathfrak{m}}$ denotes a motivic amplitude object living in a connected graded Hopf algebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ of motivic periods, with $\mathrm{per}(\mathcal{A}^{\mathfrak{m}}) = \mathcal{A}$ its numerical realization. $\mathcal{L}_G(F) = \bigoplus_{n \geq 1} \mathcal{L}_G(F)_n$ is the Goncharov Lie coalgebra of primitives, graded by weight n, with cobracket $\delta\colon \mathcal{L}_G(F)_n \to \bigoplus_{p+q=n} \mathcal{L}_G(F)_p \wedge \mathcal{L}_G(F)_q$. The master formulas of the series are

$$\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}}), \qquad \Delta(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}, \qquad \delta\colon \mathcal{L} \to \Lambda^2 \mathcal{L}, \tag{2}$$

$$\mathrm{per}(\Pi) = \int_\Gamma \omega, \qquad \Pi = (X, D, \omega, \Gamma), \qquad r_\alpha\colon K_\bullet(X) \otimes \mathbb{Q} \to H^\bullet_{\mathcal{D}}(X, \mathbb{Q}(n)). \tag{3}$$

The cobracket δ is the antisymmetrization of the reduced coproduct $\Delta'(x) := \Delta(x) - x \otimes 1 - 1 \otimes x$, landing in $\Lambda^2\mathcal{L}$ rather than $\mathcal{L} \otimes \mathcal{L}$. This paper is about the second line (3): the two arrows — period pairing and regulator — that produce numbers.

3.2 Period data and the period map

Definition 3.1 (Period datum; period map, status S)**.** A *period datum* is a quadruple $\Pi = (X, D, \omega, \Gamma)$ in which X is a smooth F-variety of dimension d, $D \subset X$ a normal-crossings divisor, $\omega \in \Omega^k_{X \setminus D}(X \setminus D)$ a closed algebraic k-form, and $\Gamma \in H_k\big(X(\mathbb{C}), D(\mathbb{C}); \mathbb{Q}\big)$ a relative singular cycle of dimension k. Its *period* is the complex number

$$\mathrm{per}(\Pi) = \int_\Gamma \omega \in \mathbb{C}. \tag{4}$$

A complex number is a *period* (in the sense of Kontsevich–Zagier) if it is the period of some datum with X, D, ω, Γ all defined over $\overline{\mathbb{Q}}$. The set of all periods forms a $\mathbb{Q}$-algebra $\mathcal{P}$, the

Kontsevich–Zagier period ring, which contains $\overline{\mathbb{Q}}$, $\log 2$, π, $\zeta(3)$, and the amplitude constants of perturbative quantum field theory.

Remark 3.2 (Periods as matrix entries of the comparison isomorphism, status S)*.* The number $\mathrm{per}(\Pi)$ is exactly a matrix entry of the comparison isomorphism comp: $H^k_{\mathrm{dR}}(X \setminus D/F) \otimes_F \mathbb{C} \xrightarrow{\sim} H^k_{\mathrm{B}}(X \setminus D, \mathbb{C})$ of the motive $M = H^k(X \setminus D)$ (equivalently, of the relative motive $H^k(X, D)$): ω names a class in the de Rham side, Γ names a class in the Betti dual, and $\int_\Gamma \omega = \langle \mathrm{comp}(\omega), \Gamma \rangle$ is the pairing of the two. This is the precise sense in which a period is the *single number* extracted from the two-sided motivic object of Rung III: de Rham (the form the calculation sees) paired against Betti (the cycle the integration runs over).

Example 3.3 (The logarithm as a Kummer period, status S)**.** Let $X = \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$, and take the relative cycle $\Gamma = [1, x]$ (a path from 1 to $x \in F^\times$) with $\omega = dt/t$. Then

$$\mathrm{per}(\Pi) = \int_1^x \frac{dt}{t} = \log x. \tag{5}$$

The underlying motive is the *Kummer motive* $M = H_1(\mathbb{G}_m, \{1, x\})$ (the relative homology; equivalently the Tate twist $H^1(\mathbb{G}_m, \{1, x\})(1)$, since $H^1(\mathbb{G}_m) = \mathbb{Q}(-1)$), a nonsplit extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$, classified by $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q}$. Thus the weight-one primitive attached to x has period $\log x$: *the raw ratios $F^\times$ of Rung I are literally the periods of the weight-one layer of the whole edifice.* In the functional-essence reading (status H), $\log x$ is a numerical shadow of the Kummer motive, and the Kummer motive is the pre-numerical source common to every way of presenting $\log x$.

3.3 Multiplicativity of the period pairing

The following is the rigorous backbone that licenses treating per as a ring homomorphism — i.e. that licenses the master formula $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ to interact correctly with amplitude factorization.

Theorem 3.4 (Additivity and multiplicativity of the period pairing, status S)**.** *Let* $\Pi_1 = (X_1, D_1, \omega_1, \Gamma_1)$ *and* $\Pi_2 = (X_2, D_2, \omega_2, \Gamma_2)$ *be period data.*

(i) (Additivity / bilinearity.) per *is $\mathbb{Q}$-bilinear in the pair (ω, Γ): for $a, b \in \mathbb{Q}$, $\int_{a\Gamma + b\Gamma'} \omega = a \int_\Gamma \omega + b \int_{\Gamma'} \omega$ and $\int_\Gamma (a\omega + b\omega') = a \int_\Gamma \omega + b \int_\Gamma \omega'$. In particular, for data sharing (X, D, ω) with disjoint cycles,* $\mathrm{per}(\Pi_1 \sqcup \Pi_2) = \mathrm{per}(\Pi_1) + \mathrm{per}(\Pi_2)$.

(ii) (Multiplicativity.) *Define the product period datum*

$$\Pi_1 \otimes \Pi_2 := \big(X_1 \times X_2,\ D_1 \times X_2 \cup X_1 \times D_2,\ p_1^*\omega_1 \wedge p_2^*\omega_2,\ \Gamma_1 \times \Gamma_2\big), \tag{6}$$

where p_j is the projection to X_j. Then

$$\mathrm{per}(\Pi_1 \otimes \Pi_2) = \mathrm{per}(\Pi_1) \cdot \mathrm{per}(\Pi_2). \tag{7}$$

Consequently the period ring $\mathcal{P}$ is closed under addition and multiplication, with the product realized by the product datum.

Proof. (i) Integration of a fixed form over a chain is $\mathbb{Q}$-linear in the chain by definition of the integral against a singular chain, and integration of a form over a fixed chain is $\mathbb{Q}$-linear in the form by linearity of the integrand; disjointness of cycles gives $\int_{\Gamma_1 \sqcup \Gamma_2} \omega = \int_{\Gamma_1} \omega + \int_{\Gamma_2} \omega$.

(ii) Write $\omega_1 \wedge \omega_2 := p_1^* \omega_1 \wedge p_2^* \omega_2$, a form on $(X_1 \setminus D_1) \times (X_2 \setminus D_2)$ of degree $k_1 + k_2 = \dim(\Gamma_1 \times \Gamma_2)$. The product cycle $\Gamma_1 \times \Gamma_2$ has complementary dimension in the product, and by Fubini's theorem for the product of chains,

$$\int_{\Gamma_1 \times \Gamma_2} p_1^* \omega_1 \wedge p_2^* \omega_2 = \Big(\int_{\Gamma_1} \omega_1 \Big) \Big(\int_{\Gamma_2} \omega_2 \Big). \tag{8}$$

This is the analytic incarnation of the Künneth decomposition $H^\bullet(X_1 \times X_2) \cong H^\bullet(X_1) \otimes H^\bullet(X_2)$ at the level of the comparison isomorphism: the period matrix of a tensor product of motives is the Kronecker product of the two period matrices, so the $(\Gamma_1 \times \Gamma_2, \omega_1 \wedge \omega_2)$ entry is the product of the (Γ_i, ω_i) entries. □

Remark 3.5 (What is and is not asserted). The nontrivial content of Theorem 3.4 is not the formal bilinearity of an integral; it is that the period ring $\mathcal{P}$ — the *image* of per — is closed under multiplication *with the product realized by an explicit product period datum.* This is what makes per a ring homomorphism rather than merely a linear functional, and it is what allows the motivic side (a Hopf algebra with a compatible product) and the numerical side (the ring $\mathcal{P}$) to be compared as *ring* objects.

Corollary 3.6 (Periods respect factorized amplitudes, status S)**.** *Suppose a physical amplitude $\mathcal{A}$ factorizes as $\mathcal{A} = \mathcal{A}_1 \cdot \mathcal{A}_2$, each factor $\mathcal{A}_i$ realized by a period datum Π_i, and suppose the joint amplitude is realized by $\Pi_1 \otimes \Pi_2$. Then the motivic identity $\mathcal{A}^{\mathfrak{m}} = \mathcal{A}_1^{\mathfrak{m}} \otimes \mathcal{A}_2^{\mathfrak{m}}$ realizes $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ compatibly with the factorization:*

$$\mathrm{per}(\mathcal{A}_1^{\mathfrak{m}} \otimes \mathcal{A}_2^{\mathfrak{m}}) = \mathrm{per}(\mathcal{A}_1^{\mathfrak{m}}) \mathrm{per}(\mathcal{A}_2^{\mathfrak{m}}) = \mathcal{A}_1 \mathcal{A}_2 = \mathcal{A}. \tag{9}$$

Proof. Immediate from Theorem 3.4(ii). □

Theorem 3.6 is the period-pairing counterpart of Rung IV's discontinuity principle (the cobracket δ predicts how an amplitude splits across a cut). Where Rung IV's δ decomposes the amplitude on the *motivic* side, Theorem 3.6 shows the *numerical* side respects the same factorization: the master formula $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ is consistent with amplitude factorization at any kinematic channel where the underlying geometry degenerates to a product.

Example 3.7 (Two familiar periods, status S)**.** Two archetypes worth naming explicitly, because they are the constants a physicist meets first. (a) $\pi = \int_{\{x^2+y^2 \le 1\}} dx\, dy$ is a period: $X = \mathbb{A}^2$, no poles, and Γ the real disc; equivalently $2\pi i = \int_{|t|=1} dt/t$ is the period of the Tate motive $\mathbb{Q}(1)$. (b) $\zeta(3) = \int_{0<t_1<t_2<t_3<1} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{1-t_3}$ is a period of $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. Both are elements of $\mathcal{P}$; both are (conjecturally) irrational realizations of pre-numerical Tate/mixed-Tate objects. They are the "numbers a calculation could in principle produce" of the dictionary — status S/H.

4 Regulator Maps

The period pairing evaluates a *single* motivic object against a *single* cycle. The regulator map is the systematic, K-theoretic packaging of this operation: it takes the whole motivic cohomology of a variety and maps it to a real-analytic target, producing not one number but a lattice of them, whose covolume is the arithmetic invariant of interest.

4.1 Motivic cohomology and algebraic K-theory

For a smooth variety X/F, Quillen's algebraic K-groups $K_m(X)$ carry a γ-filtration whose graded pieces $K_m(X)^{(n)} \otimes \mathbb{Q}$ are identified with motivic cohomology $H^{2n-m}_{\mathcal{M}}(X, \mathbb{Q}(n))$. For $X = \operatorname{Spec} F$ with F a number field, the relevant groups are governed by Borel's computation of the ranks of $K_m(F)$; the weight-n part $K_{2n-1}(F)^{(n)} \otimes \mathbb{Q} \cong H^1_{\mathcal{M}}(F, \mathbb{Q}(n))$ is the source of the regulator of interest to us. In the dictionary, a class in $H^1_{\mathcal{M}}(F, \mathbb{Q}(n))$ is a pre-numerical motivic object (status S/H); the regulator is the arrow that makes it a number.

4.2 Deligne–Beilinson cohomology

Definition 4.1 (Deligne–Beilinson cohomology, status S)**.** For a smooth projective variety $X/\mathbb{C}$ and $n \geq 0$, the Deligne complex $\mathbb{Q}_{\mathcal{D}}(n)$ is the complex of sheaves (in the analytic topology)

$$\mathbb{Q}(n) \longrightarrow \mathcal{O}_X \longrightarrow \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^{n-1}_X, \tag{10}$$

with $\mathbb{Q}(n) = (2\pi i)^n \mathbb{Q}$ in degree 0 and the truncated de Rham complex in the remaining degrees. Its hypercohomology $H^i_{\mathcal{D}}(X, \mathbb{Q}(n)) := \mathbb{H}^i(X, \mathbb{Q}_{\mathcal{D}}(n))$ is *Deligne–Beilinson cohomology*. It sits in exact sequences relating it to Betti cohomology $H^{i-1}(X, \mathbb{C}/\mathbb{Q}(n))$ and to the Hodge filtration, and it is the natural target carrying the real-analytic (archimedean) data of the motive.

The point, for the dictionary, is that $H^\bullet_{\mathcal{D}}(X, \mathbb{Q}(n))$ is a real vector space built from *both* the Betti cycle data and the de Rham form data of X — exactly the two sides whose pairing is a period. A regulator lands here precisely so that its image can be paired against cycles to give real numbers.

4.3 The regulator map

Definition 4.2 (Regulator map, general, status S)**.** A *regulator* is a natural map

$$r_{\mathcal{D}} \colon K_{2n-i}(X) \otimes \mathbb{Q} \cong H^i_{\mathcal{M}}(X, \mathbb{Q}(n)) \longrightarrow H^i_{\mathcal{D}}(X, \mathbb{Q}(n)), \tag{11}$$

from motivic cohomology to Deligne–Beilinson cohomology, compatible with products and functorial in X. Composed with a choice of topological cycle / period pairing, it produces the actual real numbers appearing in the Beilinson conjectures relating special values of L-functions to regulator determinants.

In the language of this rung, Theorem 4.2 is the concrete instance $\alpha = \mathrm{per}$ of Rung IV's abstract realization homomorphism r_α: where Rung IV's Theorem (Conditional Amplitude

Decomposition) held for an *arbitrary* Hopf-algebra homomorphism induced by a Tannakian realization functor, Rung V picks out the specific realization — the regulator/period channel — that produces honest numbers. Every other r_α (Betti, étale, Hodge) rearranges structure; only this one measures.

4.4 The Borel regulator

Definition 4.3 (Borel regulator, status S)**.** Let F be a number field with r_1 real and r_2 complex places. For $m \geq 2$, Borel's regulator is a homomorphism

$$r_B \colon K_{2m-1}(F) \otimes \mathbb{Q} \longrightarrow \mathbb{R}^{d_m}, \qquad d_m = \begin{cases} r_1 + r_2 & m \text{ odd}, \\ r_2 & m \text{ even}, \end{cases} \tag{12}$$

constructed from the Borel/van Est class in the continuous cohomology of $\mathrm{GL}_N(\mathbb{C})$ (resp. $\mathrm{GL}_N(\mathbb{R})$), applied to the stable homology of $\mathrm{GL}(\mathcal{O}_F)$.

Theorem 4.4 (Borel's rank and covolume theorem, status S)**.** *With notation as in Theorem 4.3, for each $m \geq 2$:*

(i) $\mathrm{rank}_{\mathbb{Z}} K_{2m-1}(\mathcal{O}_F) = \dim_{\mathbb{Q}} \big(K_{2m-1}(F) \otimes \mathbb{Q}\big) = d_m$*;*

(ii) r_B *is injective with image a full lattice* $\Lambda_m(F) \subset \mathbb{R}^{d_m}$*;*

(iii) *the covolume of $\Lambda_m(F)$ equals, up to an explicit nonzero rational factor, the leading Taylor coefficient $\zeta_F^*(1-m)$ of the Dedekind zeta function at $s = 1-m$:*

$$\mathrm{covol}\big(\Lambda_m(F)\big) \;\doteq\; \zeta_F^*(1-m) \qquad (\textit{mod } \mathbb{Q}^\times). \tag{13}$$

Proof (sketch; Borel 1977). Borel computes the ranks of the higher K-groups of $\mathcal{O}_F$ via the stable cohomology of $\mathrm{SL}_N(\mathcal{O}_F)$ and the Borel regulator classes, obtaining (i) and the injectivity in (ii). The covolume statement (iii) follows from comparing the Borel class to the archimedean part of the functional equation of ζ_F: the leading term at $s = 1-m$ is, up to the standard Γ-factors and a rational number, the covolume of the regulator lattice. Full details are in Borel's paper; the low-weight case $m = 2$ is the Bloch–Wigner regulator worked out completely in Section 5. □

Remark 4.5 (Why this is the paradigm of the emergent property). Theorem 4.4 is the archimedean regulator's paradigmatic theorem, and it is the cleanest general statement of this rung's emergent property. A K-theory class is a pre-numerical motivic object (Rung III/IV); the regulator r_B turns it into a point of a lattice in $\mathbb{R}^{d_m}$; and the covolume of that lattice is tied to an *independently computable* number — a special value of a zeta function. The passage "motive → measurable number" is here not a slogan but a theorem, and the number it produces ($\zeta_F^*(1-m)$) is exactly the kind of real number one could tabulate and, in physical instances, compare to data.

4.5 The Beilinson regulator and conjecture

Definition 4.6 (Beilinson regulator, status S)**.** For a smooth projective variety X/F over a number field and integers i, n, the Beilinson regulator is the map

$$r_{\mathcal{D}}\colon K_{2n-i}(X) \otimes \mathbb{Q} \longrightarrow H^i_{\mathcal{D}}(X_{/\mathbb{R}}, \mathbb{R}(n)), \tag{14}$$

the real-Deligne-cohomology-valued instance of Theorem 4.2, generalizing the Borel regulator (the case $X = \operatorname{Spec} F$).

Conjecture 4.7 (Beilinson, status S as a precise conjecture; H as a physics-generality claim)**.** Let X/F be smooth projective over a number field and i, n integers with $2n - i \neq 1$. Then $r_{\mathcal{D}}$ is an isomorphism onto a subspace of the expected dimension (the Beilinson–Soulé / non-vanishing part), and the determinant of $r_{\mathcal{D}}$ against an integral K-theory basis and a natural $\mathbb{Q}$-structure equals, up to $\mathbb{Q}^\times$, the leading Taylor coefficient of the motivic L-function $L(H^{i-1}(X), s)$ at the near-central integer $s = n$:

$$\det r_{\mathcal{D}} \;\doteq\; L^*\big(H^{i-1}(X), n\big) \qquad (\mathrm{mod}\ \mathbb{Q}^\times). \tag{15}$$

Remark 4.8 (Status discipline). Theorem 4.7 is a precise conjecture (status S as a mathematical statement), proven in a growing but still limited list of low-weight/rank-one cases — Borel's theorem (Theorem 4.4) is the case $X = \operatorname{Spec} F$; Beilinson, Deligne, and others handle Dirichlet L-values, elliptic curves in low rank, and modular forms. Its *extrapolation* to "every physical amplitude's L-value is a regulator determinant" is status H: a heuristic hope, not a theorem. Per the rule of composition, any physical conclusion drawn through Theorem 4.7 inherits status $\geq$ H.

5 The Dilogarithm, the Bloch Group, and the K_3 Regulator

This section is the analytic payoff of the whole library. The dilogarithm five-term relation, introduced in Rung II as the fundamental functional equation of Li_2 and re-read in Rung IV as a vanishing cobracket, is carried here all the way to a regulator: a concrete, fully worked map from a pre-numerical K-theory group to a single real number. It is the cleanest illustration in the series of the dictionary entry *regulator $\leftrightarrow$ passage from motivic object to physical real number.*

5.1 The five-term relation (recalled from Rung II)

Definition 5.1 (Rogers, Bloch–Wigner, and cross-ratio dilogarithms, status S)**.** The *Rogers dilogarithm* is $L(x) := \mathrm{Li}_2(x) + \frac{1}{2}\log(x)\log(1-x)$ for $0 < x < 1$. The *Bloch–Wigner function* is

$$D(z) := \operatorname{Im}\big(\mathrm{Li}_2(z)\big) + \arg(1-z)\, \log|z|, \qquad z \in \mathbb{C}, \tag{16}$$

a real-analytic, single-valued function on $\mathbb{P}^1(\mathbb{C})$, continuous on all of $\mathbb{P}^1(\mathbb{C})$, vanishing at $0, 1, \infty$, and odd under $z \mapsto \bar{z}$ and $z \mapsto 1/z$.

Proposition 5.2 (The five-term relation, three equivalent forms, status S)**.** *The following hold.*

(a) (Rogers form.) *For* $0 < x, y < 1$,

$$L(x) + L(y) = L(xy) + L\left(\tfrac{x(1-y)}{1-xy}\right) + L\left(\tfrac{y(1-x)}{1-xy}\right). \tag{17}$$

(b) (Bloch–Wigner form.) *For* $u, v \in \mathbb{C}$,

$$D(u) + D(v) + D\left(\tfrac{1-u}{1-uv}\right) + D(1-uv) + D\left(\tfrac{1-v}{1-uv}\right) = 0. \tag{18}$$

(c) (Cross-ratio / geometric form.) *For five points* $z_0, \ldots, z_4 \in \mathbb{P}^1(\mathbb{C})$ *in general position, with* $[\,\widehat{z}_i\,]$ *the cross-ratio of the four points remaining after omitting* z_i,

$$\sum_{i=0}^{4} (-1)^i \, D\big([\,\widehat{z}_i\,]\big) = 0. \tag{19}$$

Proof (sketch, standard). All three forms are equivalent by direct substitution of cross-ratios of five points into the algebraic five-term relation among Li_2-values; the geometric form follows by applying the relation to the five cross-ratios of a 5-point configuration and using the PGL_2-invariance of $D \circ$ (cross-ratio). Wojtkowiak's theorem shows every one-variable functional equation of the dilogarithm is a consequence of this single relation. See Zagier's survey [10] and [9]. □

5.2 The Bloch group

Definition 5.3 (Bloch group, status S)**.** Let F be a field. Let $\mathbb{Z}[F^\flat]$ be the free abelian group on symbols $[x]$, one for each $x \in F^\flat := F \setminus \{0, 1\}$, and let

$$\delta_2 \colon \mathbb{Z}[F^\flat] \longrightarrow \Lambda^2(F^\times), \qquad [x] \longmapsto (1-x) \wedge x. \tag{20}$$

Write $\mathcal{B}_2(F) := \ker \delta_2$ (the *Bloch elements*, on which the cobracket into $\Lambda^2 F^\times$ vanishes). The *Bloch group* is the quotient

$$\mathcal{B}(F) := \mathcal{B}_2(F) \big/ \langle \text{five-term relations} \rangle, \tag{21}$$

where the five-term relations are the elements $[x] - [y] + [\frac{y}{x}] - [\frac{1-x^{-1}}{1-y^{-1}}] + [\frac{1-x}{1-y}]$ (Suslin's normalization). Equivalently, $\mathcal{B}(F) = \mathbb{Z}[F^\flat]/\langle\text{five-term}\rangle$ intersected with $\ker \delta_2$.

Remark 5.4 (The Bloch group is the weight-2 piece of $\mathcal{L}_G(F)$, status S). The map δ_2 of Theorem 5.3 is exactly the weight-2 cobracket of Rung IV: for the motivic dilogarithm class $\mathrm{Li}_2^{\mathsf{m}}(x)$ one has $\delta\,\mathrm{Li}_2^{\mathsf{m}}(x) = (1-x) \wedge x \in \Lambda^2\big(F^\times \otimes \mathbb{Q}\big)$, since the reduced coproduct is $\Delta'\mathrm{Li}_2^{\mathsf{m}}(x) = -\mathrm{Li}_1^{\mathsf{m}}(x) \otimes \log^{\mathsf{m}}(x) = \log^{\mathsf{m}}(1-x) \otimes \log^{\mathsf{m}}(x)$ (using $\mathrm{Li}_1(x) = -\log(1-x)$), whose antisymmetrization is $\log(1-x) \wedge \log(x) = (1-x) \wedge x$. Thus the kernel $\mathcal{B}_2(F)$ is the subgroup of $\mathbb{Z}[F^\flat]$ on which the weight-2 cobracket vanishes, and $\mathcal{B}(F) \otimes \mathbb{Q}$ is, up to identification, the weight-2 part $\mathcal{L}_G(F)_2$ of the Goncharov Lie coalgebra. *The five-term relation is not an accident of* Li_2*; it is the defining relation of* $\mathcal{L}_G(F)_2$.

5.3 The Bloch–Wigner regulator

Theorem 5.5 (The Bloch–Wigner regulator on $K_3^{\mathrm{ind}}(F)$, status S)**.** *Let $F \subset \mathbb{C}$ be a field. The Bloch–Wigner function induces a well-defined group homomorphism*

$$D_*\colon \mathcal{B}(F) \longrightarrow \mathbb{R}, \qquad \textstyle\sum_j n_j[x_j] \longmapsto \sum_j n_j\, D(x_j), \tag{22}$$

and, composed with Suslin's isomorphism $K_3^{\mathrm{ind}}(F)\otimes\mathbb{Q} \cong \mathcal{B}(F)\otimes\mathbb{Q}$, yields the Bloch–Wigner regulator

$$r\colon K_3^{\mathrm{ind}}(F)\otimes\mathbb{Q} \xrightarrow{\sim} \mathcal{B}(F)\otimes\mathbb{Q} \xrightarrow{D_*} \mathbb{R}. \tag{23}$$

Its well-definedness is equivalent to D satisfying the five-term relation (18)*; i.e. to the vanishing, in $\mathcal{L}_G(F)_2$, of the cobracket combination $\sum(-1)^i D([\widehat{z}_i])$.*

Proof. By Theorem 5.4, the weight-2 part of $\mathcal{L}_G(F)$ is generated by classes $[x]$, $x \in F^\flat$, subject to exactly the relations that make the cobracket $\delta[x] = (1-x)\wedge x$ consistent; the five-term relation is the statement that a specific alternating sum of such classes lies in the kernel of the map to $\mathcal{L}_G(F)_2$, i.e. is a relation in $\mathcal{B}(F)$. The function D is real-analytic and single-valued on $\mathbb{P}^1(\mathbb{C})$, and by Theorem 5.2(b) it satisfies the five-term relation identically; hence D_* annihilates the relation subgroup and descends to a homomorphism $\mathcal{B}(F)\to\mathbb{R}$. Suslin's theorem [11] provides the isomorphism $K_3^{\mathrm{ind}}(F)\otimes\mathbb{Q}\cong\mathcal{B}(F)\otimes\mathbb{Q}$; composing gives r. □

Remark 5.6 (The single cleanest instance of the emergent property)*.* Theorem 5.5 is the library's showcase. The object being mapped, $K_3^{\mathrm{ind}}(F)$ (equivalently the weight-2 part of $\mathcal{L}_G(F)$), is *pre-numerical*: a K-theory class, an element of a coalgebra of framed motives (Rung III/IV). The regulator D is the arrow that finally *produces a real number*. And the arrow is well-defined *precisely because* of Rung II/IV's functional-equation bookkeeping: the five-term relation, which in Rung IV is the vanishing of a cobracket, is in Rung V the algebraic identity that lets a motivic K-theory class map consistently to a single real number. The cobracket δ ("factorization channel / information split" in the dictionary) is exactly the obstruction whose vanishing licenses measurement.

5.4 The number the machine outputs

The whole point of a regulator is that it emits a *number*. We exhibit three.

Example 5.7 ($D(i)$ is Catalan's constant, status S)**.** Take $z = i$. Since $|i| = 1$ we have $\log|i| = 0$, so the second term of D vanishes and

$$D(i) = \operatorname{Im}\bigl(\mathrm{Li}_2(i)\bigr) = \sum_{k\geq 0}\frac{(-1)^k}{(2k+1)^2} = G = 0.9159655941\ldots, \tag{24}$$

Catalan's constant. Indeed $\mathrm{Li}_2(i) = -\frac{\pi^2}{48} + iG$, so the imaginary part is G exactly. This is a genuine, checkable real number produced by the regulator on the weight-2 motivic class $[i]\in\mathcal{B}(\mathbb{Q}(i))$.

Example 5.8 (The maximum of D and the regular ideal tetrahedron, status S)**.** On the upper half-plane D attains its maximum at the sixth root of unity $z = e^{i\pi/3} = \frac{1+i\sqrt{3}}{2}$, with

$$D\big(e^{i\pi/3}\big) = \mathrm{Im}\ \mathrm{Li}_2\big(e^{i\pi/3}\big) = 1.0149416064\ldots, \tag{25}$$

which is exactly the hyperbolic volume of the *regular ideal tetrahedron*. The figure-eight knot complement decomposes into two such tetrahedra, so its volume is $2\,D(e^{i\pi/3}) \approx 2.02988\ldots$. Here the regulator's output is literally a *measurable geometric quantity* — a hyperbolic volume — and, via Borel's theorem (Theorem 4.4) specialized to $m = 2$ and $F = \mathbb{Q}(\sqrt{-3})$, this same number is (up to an explicit rational and powers of π) the special value $\zeta_F(2)$. This is the tightest possible knot of the whole ladder: one real number is simultaneously a dilogarithm value (Rung II), a Bloch-group regulator (Rung V), a hyperbolic volume (geometry), and a zeta special value (arithmetic).

Example 5.9 (A rational multiple in disguise, status S)**.** For a real argument, D degenerates (D vanishes on $\mathbb{P}^1(\mathbb{R})$), but the underlying Li_2 still emits recognizable periods: $\mathrm{Li}_2(1) = \zeta(2) = \pi^2/6$ and $\mathrm{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2$. These are the "rational-multiple" boundary of the regulator: weight-2 classes whose regulator collapses to a period times a rational, reflecting that the corresponding K_3-class is torsion or defined over $\mathbb{Q}$ (where $K_3^{\mathrm{ind}}(\mathbb{Q}) \otimes \mathbb{Q} = 0$). They mark exactly where "motive $\to$ number" produces something already visible in Rung I.

5.5 The cobracket–regulator square

We can now display the exact commuting relationship between Rung IV's cobracket and Rung V's regulator, which is the structural heart of this paper.

$$\begin{tikzcd}
K_3^{\mathrm{ind}}(F) \otimes \mathbb{Q} \arrow[r, "\sim"] \arrow[d, "r"'] & \mathcal{L}_G(F)_2 \arrow[d, "\delta"] \\
\mathbb{R} & \Lambda^2\big(F^\times \otimes \mathbb{Q}\big) \arrow[l, "?"']
\end{tikzcd}$$

The top row is Suslin's identification (Theorem 5.5); the right arrow is Rung IV's weight-2 cobracket $\delta[x] = (1-x) \wedge x$; the left arrow is Rung V's regulator $r = D_*$. The bottom arrow marked "?" does not exist as a map of the displayed groups — and that is the point. The regulator r is defined on the *kernel* of the cobracket (the Bloch elements $\mathcal{B}_2(F) = \ker \delta_2$): D produces a well-defined number *exactly on those classes whose cobracket vanishes*. The cobracket detects the "information split" of a weight-2 observable; the regulator measures precisely the part that does not split further. *Measurement is the evaluation of the primitives that survive the decomposition law.*

6 Physical Interpretation via the Dictionary

We collect the physical readings, each with its S/H/P label, and add the decomposition semantics that this rung contributes to the series.

Mathematics	Physical representation	Decomposition meaning	Status
Period $\int_\Gamma \omega$	Measured number from geometry	Value of a fully evaluated observable	S
Form ω (de Rham)	Integrand seen by calculation	The analytic "what"	S
Cycle Γ (Betti)	Integration domain / history	The topological "where"	S
Comparison pairing	Two calculational routes agree	Same number, two computations	S
Regulator r_α	Measurement / numerical extraction	Evaluation of surviving primitives	S/H
Borel regulator	Archimedean measurement of K-class	Lattice covolume $\to$ $\zeta_F^*(1-m)$	S
Bloch–Wigner D	Weight-2 measured number	Regulator of a single cut-free primitive	S
Bloch group $\mathcal{B}(F)$	Space of weight-2 primitives	ker(cobracket): what does not split	S
Five-term relation	Consistency of measurement	Vanishing cobracket = well-defined number	S
Period ring $\mathcal{P}$	Reachable physical constants	The measurable output space	S/H
Beilinson conjecture	L-value = regulator determinant	Arithmetic $\leftrightarrow$ analytic bridge	H (gen.)
Regulator output $\in F^\times$	New measured scale	Re-entry into Rung I (loop closure)	S/H

The last line is this rung's specific contribution to the series' thesis. The regulator's output is a real number, and a real number (a measured scale, ratio, or cross-ratio) is exactly an element of Rung I's domain $F^\times$. Measurement is therefore not an exit from the mathematical structure but a *re-entry* into its ground floor: the ladder closes into a loop. This is the hand-off to the synthesis rung.

7 Discussion

7.1 Limitations

We reproduce the series' limitation discipline, specialized to this rung.

1. **Not every analogy is a theory.** Theorems 3.4, 4.4 and 5.5 are status S as mathematics; their universal *physical* reading (regulator = measurement of any observable) is status H. The S/H/P labels are part of the formalism.

2. **The Beilinson conjecture is open in general.** Beyond low-weight/rank-one cases (Borel, Dirichlet, some elliptic and modular cases), Theorem 4.7 is unproven. Any physical conclusion routed through it inherits status $\geq$ H by the rule of composition.

3. **The Non-Tate obstruction limits the tame story.** When a motivic amplitude lift has the H^1 of an elliptic curve (or another non-Tate simple motive) as a subquotient, the polylogarithmic anatomy of Rung IV is unavailable and the period matrix contains genuine elliptic periods ω_1, ω_2 (Legendre relation $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$), not $\mathbb{Q}$-combinations of multiple zeta values. The correct regulator then lives on an *elliptic*-polylogarithm coalgebra; the Bloch–Wigner regulator of Theorem 5.5 is the tame, weight-2, mixed-Tate case only.

4. **p-adic and syntomic regulators are a parallel story.** The archimedean regulators of this paper have non-archimedean analogues (Besser's syntomic regulator, Nekovář's p-adic Beilinson conjectures) that measure the same K-theory classes against p-adic rather than real periods. They are outside our scope but are a natural extension of the "motive $\to$ number" arrow to p-adic numbers.

7.2 A novel comparison: cuts versus regulators

Rung IV showed (its Proposition on discontinuities) that the discontinuity of an amplitude across a branch point associated to a weight-one primitive $\log s$ picks out exactly the coaction terms whose second slot is $\log^{\mathfrak{m}} s$, with coefficient $2\pi i$. This is a *cut*: an operation that lowers weight by extracting a coaction slot. A regulator, by contrast, is an operation that *evaluates* a class against a cycle to produce a number. To this KB's knowledge, no systematic comparison exists in the literature between the informal "cut/discontinuity" operations of the amplitudes community and the Beilinson/Borel regulator machinery of arithmetic geometry. The cobracket–regulator square of Section 5.5 is a first step: it shows that on the weight-2 nose, the cut (cobracket $\delta[x] = (1-x) \wedge x$) and the measurement (regulator D_*) are defined on complementary parts of the same group — the cobracket on the whole of $\mathbb{Z}[F^\flat]$, the regulator on its kernel $\mathcal{B}_2(F)$. A precise general statement — that "taking a discontinuity" and "applying a regulator" are adjoint operations on the motivic coaction — is, we believe, a genuinely open and physically motivated problem.

7.3 Open problems

- Extend the regulator/Beilinson formalism to the elliptic and modular non-Tate regime obstructed above: what is the correct "regulator" on the elliptic-polylogarithm coalgebra, and does an elliptic analogue of the Bloch–Wigner five-term consistency hold?
- Make precise the cut-versus-regulator adjunction of Section 7.2.
- Determine, for the recently reconstructed Goncharov Lie coalgebra [18], the exact weight-3 (and higher) analogues of the Bloch–Wigner regulator, i.e. the higher-weight measurement maps out of $K_5^{(3)}(F)$.

7.4 Loop closure toward the synthesis

The composite of the five rungs,

$$F, F^{\times} \xrightarrow{\mathrm{I}} \mathrm{Li}_n, \text{iter. int.} \xrightarrow{\mathrm{II}} \mathrm{Mot}(X) \xrightarrow{\mathrm{III}} \Delta, \delta, \mathcal{L}_G(F) \xrightarrow{\mathrm{IV}} \text{per, regulator} \xrightarrow{\mathrm{V}} \text{observable}, \tag{26}$$

reproduces the master formula observable = per(motive), with each rung supplying exactly one arrow. This paper supplied the last arrow — the one that produces a number. The number produced by Rung V (a $\log x$, a $D(z)$, a Borel covolume) is itself an element of Rung I's domain $F^{\times}$: a measured scale one could feed back into a new kinematic configuration. Measurement is therefore *re-entry*, not exit; the ladder is a loop. The synthesis rung recomposes (26) and tabulates the S/H/P statistics of the whole; this paper's contribution to that synthesis is the rigorous arrow per (Theorem 3.4) and its worked, number-emitting instance r (Theorem 5.5).

8 Formalization

The theory is accompanied by machine-checkable artifacts, in the spirit of the series' commitment to formal transparency. We summarize them; the full sources are distributed with the paper.

8.1 Haskell

The Haskell module `Regulators` provides:

- a `PeriodDatum` type (X, D, ω, Γ) and a symbolic period algebra witnessing the additivity/multiplicativity of Theorem 3.4 (the product datum realizes the product of periods);
- a numerical Bloch–Wigner function `blochWigner :: Complex Double -> Double` via a series-plus-reflection evaluation of Li_2, together with a numerical check of the five-term relation (18) and the identity $D(i) = G$ (Catalan) of Theorem 5.7;
- a representation of Bloch-group elements $\sum n_j[x_j]$, the weight-2 cobracket $[x] \mapsto (1-x) \wedge x$ into $\Lambda^2 F^{\times}$, and a check that the five-term combination lies in its kernel (Theorem 5.4);
- the Borel rank function d_m of Theorem 4.3.

8.2 Lean

The Lean file `Regulators.lean` (self-contained, no `Mathlib` import, namespace `Regulators`) declares the core structures — `PeriodDatum`, the period map, the `Regulator` map, and the `BlochGroup` as a quotient by the five-term relation — and states the main theorems: multiplicativity of the period pairing (Theorem 3.4), the Borel rank formula (Theorem 4.4(i)), and well-definedness of the Bloch–Wigner regulator (Theorem 5.5). Statements provable from the elementary scaffolding are proved; the deep theorems (Suslin's isomorphism, Borel's covolume) are recorded as `theorem` statements with `sorry` and a comment naming the paper result, in the honest spirit of the S/H/P discipline.

9 Conclusion

We built the arrow the ladder was missing. Rungs I–IV assemble a pre-numerical machine: a disciplined domain of ratios, the transcendental functions on it, the motive behind those functions, and the coalgebraic law that anatomizes the motive. Rung V turns the crank. The period pairing $\mathrm{per}(\Pi) = \int_\Gamma \omega$ pairs de Rham form against Betti cycle to produce a number, and Theorem 3.4 shows this pairing is a ring homomorphism compatible with amplitude factorization. The regulator map r_α packages the pairing K-theoretically; Borel's theorem (Theorem 4.4) ties its output lattice to Dedekind zeta values, and Beilinson's conjecture (Theorem 4.7) is the general L-value template. The showcase is the Bloch–Wigner regulator (Theorem 5.5): the dilogarithm five-term relation of Rung II, re-read as a vanishing cobracket in Rung IV, is exactly what makes a K_3-class map consistently to a single real number — and that number is Catalan's constant, or a hyperbolic volume, or a zeta value. The emergent property is the passage from motivic object to measurable number; and because the number re-enters Rung I's $F^\times$, the ladder closes into a loop. Measurement, in this library, is the evaluation of the primitives that survive the decomposition law.

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