Polylogarithms and Iterated Propagation: Transcendental Weight as Loop Order

Rung II of a Modular Ladder from Kinematic Domains to Measurement

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Abstract

We develop the second rung of a modular library that reads a Goncharov-style Lie coalgebra as a *bookkeeping system* for how physical observables decompose into irreducible informational constituents. The governing perspective treats a physical observable — a scattering amplitude, a Feynman integral, a period — as the numerical realization of a pre-numerical structured object, and asks not *what is its value* but *what is it made of.* Rung I supplied the disciplined domain of observable ratios: a field $\mathbb{F}$, its multiplicative group $\mathbb{F}^\times$, cross-ratios, and the punctured projective line $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. This paper introduces the graded family of special functions defined on that domain: the classical polylogarithm Li_n, the multiple polylogarithms, multiple zeta values, and Chen's iterated path integrals, organized by a *weight* and *depth* bigrading. The emergent structural property of this rung is a precise slogan: **transcendental weight = loop order**. We prove Chen's path-independence theorem for iterated integrals of logarithmic forms, we establish that the iterated reduced coproduct of a weight-n polylogarithm terminates after at most $n-1$ steps at weight-one logarithmic leaves, and we give a complete, three-form treatment (Rogers, Bloch–Wigner, cross-ratio) of the five-term relation for the dilogarithm — the single load-bearing worked example that threads through the whole ladder, reappearing as the defining relation of the Bloch group (a later rung's cobracket kernel) and as the well-definedness condition for the Bloch–Wigner regulator. Every entry of the math↔physics dictionary and every claim is tagged with an epistemic status label S/H/P (standard / heuristic / speculative), and status composes monotonically. Accompanying Haskell code numerically certifies the shuffle product, the symbol of Li_2, and all three forms of the five-term relation; a Lean 4 file states the rung's structural theorems. The library is standalone: all citations are to the external literature.

Contents

1 Introduction

1.1 The governing perspective

The organizing idea of this library is a single sentence, which we ask the reader to keep in view throughout:

A Goncharov-style Lie coalgebra is not merely a formal algebraic object; it is a *book-keeping system* for how physical quantities break apart into irreducible informational constituents.

A physical observable — a scattering amplitude, a Feynman integral, a period, an entropy — is treated as the *numerical realization* of a deeper, pre-numerical, structured object. The question "what is this observable made of?" is then answered not by further computation but by a *decomposition law*: a coproduct/cobracket that exposes the observable's primitive, irreducible pieces (cuts, discontinuities, factorization channels). The full library builds this apparatus rung by rung and, in its final synthesis, identifies *measurement* as the closure of the whole ladder.

This is a *modular* framework, not a unified one. Each rung is a standalone, defensible development; each composes with its neighbour to add exactly one emergent structural property that the level below cannot express on its own. The ladder is

$$\underbrace{\mathrm{I}}_{\substack{\text{kinematic}\\\text{domains}}} \longrightarrow \underbrace{\mathrm{II}}_{\substack{\text{polylogarithms}\\\text{(this paper)}}} \longrightarrow \underbrace{\mathrm{III}}_{\text{motives}} \longrightarrow \underbrace{\mathrm{IV}}_{\substack{\text{Lie}\\\text{coalgebra}}} \longrightarrow \underbrace{\mathrm{V}}_{\text{regulators}} \longrightarrow \underbrace{\text{Synthesis}}_{\text{measurement}}.$$

1.2 Where this rung sits

Rung I disciplined the raw arena of observation: a field $\mathbb{F}$ of kinematic invariants, its multiplicative group $\mathbb{F}^\times$ of nonzero scales and ratios, the projective line $\mathbb{P}^1(\mathbb{F})$ with its $\mathrm{PGL}_2(\mathbb{F})$ action, the cross-ratio, and the moduli space $\mathrm{Conf}_n(\mathbb{P}^1) = M_{0,n}$. Its central computation showed that the four-point kinematic configuration space is *exactly* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ — precisely the punctured domain on which the classical polylogarithm is single-valued away from its branch points at $0, 1, \infty$. Rung I therefore does not merely supply "some field"; it hands over the *specific* domain on which the functions of this paper live.

Rung II — the present paper — populates that domain with functions. We introduce the classical polylogarithm $\mathrm{Li}_n(x) = \sum_{k\geq 1} x^k/k^n$, its multi-variable generalizations, the multiple zeta values, and their common analytic incarnation as Chen iterated path integrals. These carry a *weight* and a *depth* grading. The emergent property we name is the slogan

$$\boxed{\textbf{transcendental weight} = \textbf{loop order}}$$

the statement that the algebraic complexity (weight) of the special functions produced by a perturbative computation tracks the physical complexity (loop order, number of propagators) of that computation. (The ladder sometimes calls this "transcendental depth," meaning the transcendentality level; in this paper we reserve the word *depth* for the distinct iterated-integral nesting grading of Theorem 6.1, and use *weight* for the grading that tracks loop order.) This is the first appearance in the ladder of a *graded* family of transcendental functions whose grading is physically meaningful, and it is the structural precondition for everything that follows: the coproduct of Rung IV lowers weight, the periods of Rung III are the weight-graded transcendental constants, and the regulators of Rung V evaluate weight-two data through the Bloch–Wigner function.

1.3 What we prove

The paper is expository where it must recall Rung I and forward-reference Rungs III–V, but it contains genuine proofs of the results that belong to Rung II proper:

- Theorem 7.1 (Chen path-independence): iterated integrals of closed logarithmic 1-forms depend only on the based homotopy class of the integration path, so that Li_n is a well-defined multivalued analytic function on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

- Theorem 7.3 (termination of the iterated coproduct): the reduced coproduct strictly lowers weight in each tensor factor, so its iteration on a weight-n polylogarithm terminates after at most $n-1$ steps at weight-one logarithmic leaves. This is the precise fact that licenses the phrase "weight = loop order": complexity is bounded, finite, and reduces to elementary log data.
- Theorem 9.2 (the five-term relation, three forms): the fundamental functional equation of Li_2 in Rogers, Bloch–Wigner, and cross-ratio forms, with the equivalences and the descent to a 5-point configuration proved in detail. This is the connective tissue of the whole library.
- Theorem 7.5 (symbol and shuffle): the symbol of Li_2 is $\mathcal{S}(\mathrm{Li}_2(x)) = -(1-x)\otimes x$, and the shuffle relations express equivalent orderings of iterated propagation.

The accompanying HASKELL package numerically certifies the shuffle product, the symbol of the dilogarithm, and all three forms of the five-term relation to machine precision; a LEAN 4 file records the structural statements. Section 16 describes both.

1.4 Standalone-library discipline and epistemic honesty

This library is self-contained: it cites only the external mathematical and physical literature (Section 20), and it makes its interpretive commitments explicit through a three-valued status system, described in Section 3, that tags every dictionary entry and every claim as *standard* (S), *heuristic* (H), or *speculative* (P). We regard this discipline as part of the formalism, not as decoration: it is what keeps the physical reading of a theorem from being silently promoted above the reliability of the mathematics that supports it.

2 The Mathematics ↔ Physics Dictionary

Every paper in this library reproduces, verbatim, the following translation table as its opening dictionary. It is the seed from which each rung's topic-specific refinements are drawn.

Mathematics	**Physical representation**
Field $\mathbb{F}$	Kinematic/coordinate domain of possible values
Element of $\mathbb{F}^\times$	Ratio, cross-ratio, scale, observable parameter
Polylogarithm $\mathrm{Li}_n(x)$	Iterated propagation / layered amplitude contribution
Period	Measurable physical number obtained from geometry
Motive	Pre-numerical informational object behind the measurement
Lie algebra	Infinitesimal symmetry / generator structure
Lie coalgebra	Decomposition law of observables
Cobracket δ	Factorization channel, cut, boundary, or information split
Weight grading	Complexity depth / loop order / transcendental depth
Primitive element	Irreducible informational unit

Coproduct	Full hierarchical decomposition
Regulator map	Passage from motivic object to physical real number
Hodge/de Rham realization	Analytic form seen by calculation
Betti realization	Topological/path/integration-cycle form
Period pairing	Physical observable as pairing of form and cycle

This seed table is reproduced verbatim across the library. One word deserves a caveat local to the present rung: in the "Weight grading" row the phrase "transcendental depth" uses *depth* in its colloquial sense of *transcendentality level* (a synonym for weight). Throughout the body of this paper we instead reserve the technical term *depth* for the iterated-integral nesting grading of Theorem 6.1, and use *weight* for the grading that tracks loop order.

Four central slogans recur across the whole library:

observable = realization(motivic information),

coalgebra = grammar of physical decomposition,

Physics observes numerical shadows of deeper motivic structures,

Physical reality is the realization layer of structured mathematical information.

3 The Epistemic Status System (S / H / P)

Every dictionary entry and every claim below carries one of three labels. These are the library's core epistemic-honesty device.

Label	**Meaning**	**Canonical example**
S	Standard use in mathematics or mathematical physics	The Goncharov coproduct on motivic polylogarithms
H	Strong heuristic representation-dictionary entry	"Weight = loop order"; cobracket as "cut"
P	Speculative philosophical ontology	Physical reality as motivic-categorical realization

Composite labels S/H and H/P occur where an entry is standard as pure mathematics but heuristic or speculative in its *physical* reading. We adopt the following rule without exception.

Definition 3.1 (Status monotonicity)**.** Order the labels by $\mathsf{S} < \mathsf{H} < \mathsf{P}$ (increasing speculativeness, equivalently decreasing reliability). If two representation entries or theorems compose, the status of the composite is the *maximum* of the two under this order: a chain of translations is only as reliable as its weakest link. In particular, a P-labeled step cannot be upgraded to S merely by composing it with otherwise-rigorous results.

We flag the reading "weight = loop order" as S/H: the weight grading and its properties are standard mathematics (S), while its identification with physical loop order is a strong heuristic (H) that is established case-by-case rather than by a universal theorem. We will be explicit about where the boundary lies (Section 14).

4 Master Formulas

The whole library shares the following notation.

$$\text{observable} = \text{Real(abstract structure)}, \tag{1}$$

$$\mathcal{A} = \text{per}(\mathcal{A}^{\mathfrak{m}}), \tag{2}$$

$$\Delta(\mathcal{A}^{\mathfrak{m}}) = \textstyle\sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2}, \tag{3}$$

$$\delta : \mathcal{L}_G \longrightarrow \Lambda^2 \mathcal{L}_G, \tag{4}$$

$$\text{wt}(x) = n \ \text{ for } x \in \mathcal{L}_{Gn}, \qquad \text{dep}(x) = \text{depth}, \tag{5}$$

$$\text{per}(\Pi) = \int_\Gamma \omega, \qquad \Pi = (X, D, \omega, \Gamma). \tag{6}$$

Here $\mathbb{F}$ is a field (typically a number field or a field of kinematic invariants), $\mathbb{F}^\times$ its multiplicative group; $\mathcal{A}^{\mathfrak{m}}$ is a *motivic amplitude object*, a pre-numerical element of a Hopf algebra $\mathcal{H}$ of motivic periods, and per the period map sending it to the physical number $\mathcal{A}$; $\mathcal{L}_G = \mathcal{L}_G(\mathbb{F}) = \bigoplus_{n\geq 1} \mathcal{L}_G(\mathbb{F})_n$ is the (Goncharov) Lie coalgebra of primitives, graded by weight n; and δ is the antisymmetrization of the reduced coproduct Δ' (namely Δ minus its two trivial terms $x \otimes 1$ and $1 \otimes x$), landing in $\Lambda^2 \mathcal{L}_G$ rather than $\mathcal{L}_G \otimes \mathcal{L}_G$. Rung II is where the weight grading of (5) first appears and where the functions realized by per in (2) are constructed.

5 Core Objects: Polylogarithms and Iterated Integrals

5.1 The classical polylogarithm

Definition 5.1 (Classical polylogarithm; status S)**.** For $n \geq 1$ the *classical polylogarithm* is the power series

$$\text{Li}_n(x) \ = \ \sum_{k\geq 1} \frac{x^k}{k^n}, \qquad |x| < 1, \tag{7}$$

continued analytically to $\mathbb{C} \setminus [1, \infty)$. The first two cases are

$$\text{Li}_1(x) = -\log(1-x), \qquad \text{Li}_2(x) = \sum_{k\geq 1} \frac{x^k}{k^2},$$

and the members are linked by the differential relations

$$\frac{\mathrm{d}}{\mathrm{d}x} \ \text{Li}_n(x) = \frac{\text{Li}_{n-1}(x)}{x} \ (n \geq 2), \qquad \frac{\mathrm{d}}{\mathrm{d}x} \ \text{Li}_1(x) = \frac{1}{1-x}. \tag{8}$$

Equivalently, $\text{Li}_n(x) = \int_0^x \text{Li}_{n-1}(t) \, \frac{\mathrm{d}t}{t}$ for $n \geq 2$, expressing Li_n as a single integration of Li_{n-1} against the logarithmic kernel $\mathrm{d}t/t$.

Iterating (8) down to Li_1 realizes Li_n as an n-fold iterated integral, which we make precise in Theorem 5.4 below. The number n is the *weight*; it counts the layers of integration.

5.2 Multiple polylogarithms and multiple zeta values

Definition 5.2 (Multiple polylogarithm; status S)**.** For positive integers $n_1, \ldots, n_d$ and complex arguments $x_1, \ldots, x_d$,

$$\text{Li}_{n_1,\ldots,n_d}(x_1, \ldots, x_d) \ = \sum_{0<k_1<\cdots<k_d} \frac{x_1^{k_1} \cdots x_d^{k_d}}{k_1^{n_1} \cdots k_d^{n_d}}, \tag{9}$$

convergent for $|x_i| \leq 1$ with $(n_d, x_d) \neq (1, 1)$. Its *weight* is $n = n_1 + \cdots + n_d$ and its *depth* is d.

Definition 5.3 (Multiple zeta values; status S)**.** The specialization to all arguments equal to 1,

$$\zeta(n_1,\ldots,n_d) \;=\; \mathrm{Li}_{n_1,\ldots,n_d}(1,\ldots,1) \;=\; \sum_{0<k_1<\cdots<k_d} \frac{1}{k_1^{n_1}\cdots k_d^{n_d}}, \qquad n_d \geq 2, \tag{10}$$

is a *multiple zeta value* (MZV) of weight $n = n_1 + \cdots + n_d$ and depth d. These are the special constants that appear as the weight-graded pieces of loop amplitude expansions.

5.3 Chen iterated integrals

The uniform analytic home of all of the above is Chen's calculus of iterated path integrals.

Definition 5.4 (Chen iterated integral; status S)**.** Fix points $a_1,\ldots,a_N \in \mathbb{C}$ (the *singularities*) and endpoints a_0, a_{N+1}. For a path $\gamma : [0,1] \to \mathbb{C} \setminus \{a_1,\ldots,a_N\}$ from a_0 to a_{N+1}, the *iterated integral* of the logarithmic forms $\omega_i = \mathrm{d}t/(t-a_i)$ is the ordered-simplex integral

$$I(a_0;a_1,\ldots,a_N;a_{N+1}) \;=\; \int_{0<t_1<\cdots<t_N<1} \frac{\gamma'(t_1)\,\mathrm{d}t_1}{\gamma(t_1)-a_1}\cdots\frac{\gamma'(t_N)\,\mathrm{d}t_N}{\gamma(t_N)-a_N}. \tag{11}$$

Here the simplex is oriented $0 < t_1 < \cdots < t_N < 1$, so t_1 is the innermost variable and the leftmost letter a_1 is the one integrated first. When an endpoint coincides with a singularity the integral is defined by tangential base-point regularization (Deligne; Goncharov). In Goncharov's convention the classical polylogarithm is the special value

$$\mathrm{Li}_n(x) \;=\; -\,I(0;1,\underbrace{0,\ldots,0}_{n-1};x), \tag{12}$$

the iterated integral along the straight path from 0 to x with the alphabet $\{\mathrm{d}t/t,\ \mathrm{d}t/(t-1)\}$: the innermost step is taken against the puncture at 1, the remaining $n-1$ steps against the puncture at 0.

Example 5.5 (Li_2 as a double iterated integral; status S)**.** We verify (12) for $n = 2$ directly. By definition, with the innermost letter $a_1 = 1$ and outer letter $a_2 = 0$,

$$-I(0;1,0;x) = -\int_{0<t_1<t_2<x} \frac{\mathrm{d}t_1}{t_1-1}\,\frac{\mathrm{d}t_2}{t_2} = -\int_0^x \frac{\mathrm{d}t_2}{t_2}\int_0^{t_2}\frac{\mathrm{d}t_1}{t_1-1}.$$

The inner integral is $\int_0^{t_2} \mathrm{d}t_1/(t_1-1) = \log(1-t_2)$, so

$$-I(0;1,0;x) = -\int_0^x \frac{\log(1-t_2)}{t_2}\,\mathrm{d}t_2 = \mathrm{Li}_2(x),$$

using $\mathrm{Li}_2(x) = -\int_0^x \log(1-t)\,\mathrm{d}t/t$ from (8). This makes the "iterated propagation" picture concrete: each of the two integrations is one propagation step, the innermost against the puncture at 1 (the form $\mathrm{d}t/(t-1)$, contributing $\log(1-t)$) and the outer against the puncture at 0 (the form $\mathrm{d}t/t$). The same computation with $n-1$ trailing $\mathrm{d}t/t$ factors reproduces $\mathrm{Li}_n(x) = \sum_{k\geq1} x^k/k^n$ by raising the exponent once per outer integration.

The passage (12) is the bridge between the "propagation" picture (each factor $\mathrm{d}t/(t-a_i)$ is one elementary propagation step against a singular locus) and the combinatorial machinery — shuffle products, the coproduct, the symbol — that organizes the whole weight-graded family. We record the two products that iterated integrals satisfy.

Proposition 5.6 (Shuffle product; status S)**.** *Iterated integrals along a fixed path multiply by the shuffle product:*

$$I(a_0;\,\vec{u}\,;a_{N+1})\cdot I(a_0;\,\vec{v}\,;a_{N+1}) \;=\; \sum_{\sigma\in\vec{u}\,\sqcup\!\sqcup\,\vec{v}} I(a_0;\,\sigma\,;a_{N+1}), \tag{13}$$

the sum running over all interleavings σ of the two letter sequences $\vec{u},\vec{v}$ preserving the internal order of each. Physically (status H) the shuffle records that two sequences of propagation steps performed along the same path may be interleaved in all order-preserving ways without changing the accumulated value.

Proof. Expand the product of the two simplex integrals over the common interval $[0,1]$. The product of an r-simplex and an s-simplex integral is an integral over $[0,1]^{r+s}$ intersected with the two independent orderings; decomposing $[0,1]^{r+s}$ into the $\binom{r+s}{r}$ chambers on which a single total order of the $r+s$ times holds, and discarding the measure-zero coincidence loci, expresses the product as the sum over all interleavings preserving the internal order of each factor. Each chamber is exactly one shuffle term (13). □

The series representation of Theorem 5.3 satisfies a second, distinct product (the *stuffle* or quasi-shuffle), obtained by splitting the summation region $\{k_1 < \cdots\} \times \{l_1 < \cdots\}$ into the regions where indices are $<$, $>$, or $=$. The two products together impose the double-shuffle relations among MZVs; we use only the shuffle relation computationally (Section 16).

5.4 The symbol

Definition 5.7 (Symbol; status S)**.** The *symbol* of a weight-n (multiple) polylogarithm F is the length-n tensor $\mathcal{S}(F) \in (\mathbb{F}^\times \otimes \mathbb{Q})^{\otimes n}$ obtained by maximally iterating the reduced coproduct down to weight-one letters. Concretely it is computed recursively from the differential: if $\mathrm{d}F = \sum_i F_i \,\mathrm{dlog}\, R_i$ with $R_i \in \mathbb{F}^\times$ and each F_i of weight $n-1$, then

$$\mathcal{S}(F) \;=\; \sum_i \mathcal{S}(F_i)\otimes R_i. \tag{14}$$

The symbol is the sharpest elementary invariant for detecting functional identities among polylogarithms: two weight-n combinations are equal modulo lower-weight products of logarithms and constants iff their symbols agree.

6 Weight, Depth, and the Emergent Property

Definition 6.1 (Weight and depth grading; status S/H)**.** The *weight* $n = n_1+\cdots+n_d$ measures transcendental complexity; the *depth* d counts the number of nested iterated-integration layers. The physical reading (status H) is:

$$\begin{aligned} \text{weight} &= \text{loop order} \;/\; \text{functional transcendentality},\\ \text{depth} &= \text{number of independent kinematic channels}. \end{aligned}$$

The weight grading is compatible with all the structure introduced so far: the differential (8) lowers weight by one; the shuffle product (13) is weight-additive; the symbol (14) of a weight-n function is a length-n tensor of weight-one letters. The grading is what makes "how complicated is this function" into a number rather than an impression, and the emergent property of the rung is the physical interpretation of that number.

Remark 6.2 (The emergent property: transcendental weight = loop order; status S/H)**.** Perturbative computations in quantum field theory produce, order by order in the loop expansion, transcendental functions of increasing weight: a one-loop finite integral is generically weight two (dilogarithms), an ℓ-loop integral is generically weight 2ℓ in four dimensions, and the constants appearing are MZVs of the corresponding weight. The empirical regularity that weight tracks loop order is the emergent property this rung names. It is *not* a theorem in the generality of all QFTs: it is a strong heuristic (H) with a large body of confirming case studies (finite integrals in ϕ^4 theory; the six-gluon remainder function in planar $\mathcal{N} = 4$ super Yang–Mills; the sunrise family up to its elliptic boundary, Section 14) and known exceptions once elliptic and higher-genus geometry enters. The mathematical content that *is* a theorem — and that makes the heuristic precise and bounded — is the termination statement of the next section: complexity, whatever its physical origin, is finitely generated by weight-one logarithms.

7 Chen's Theorem and Termination of the Coproduct

7.1 Path-independence

Theorem 7.1 (Chen path-independence for logarithmic forms; status S)**.** *Let $a_1, \ldots, a_N \in \mathbb{C}$ and let $\gamma_0 \simeq \gamma_1$ be homotopic paths in $\mathbb{C} \setminus \{a_1, \ldots, a_N\}$ with the same endpoints, the homotopy fixing the endpoints. Then the iterated integral (11) of the closed logarithmic forms $\omega_i = \mathrm{d}t/(t - a_i)$ is unchanged:*

$$I_{\gamma_0}(a_0; a_1, \ldots, a_N; a_{N+1}) \;=\; I_{\gamma_1}(a_0; a_1, \ldots, a_N; a_{N+1}),$$

with tangential regularization at any singular endpoint. Consequently the classical polylogarithm

$$\mathrm{Li}_n(x) \;=\; -\,I(0; 1, 0, \ldots, 0; x)$$

of (12) is a well-defined multivalued analytic function on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ whose monodromy is exactly the analytic-continuation data organized by the later cobracket.

Proof. This is Chen's fundamental theorem of iterated path integrals (Chen, 1977). The key point is that the iterated integral of a collection of *closed* 1-forms along a path is invariant under homotopy of the path rel endpoints. We sketch the standard chain-homotopy argument. Let $H : [0,1] \times [0,1] \to \mathbb{C} \setminus \{a_1, \ldots, a_N\}$ be a homotopy with $H(\cdot, 0) = \gamma_0$, $H(\cdot, 1) = \gamma_1$, and $H(0, s), H(1, s)$ constant in s. Regard the length-N iterated integral as the evaluation of a 1-form Θ_N on the based path space, built from the forms ω_i; Chen's transport formula gives $\mathrm{d}\Theta_N = \sum \pm \Theta_k \wedge \Theta'_{N-k}$ with each factor proportional to $\mathrm{d}\omega_i = 0$, because every ω_i is closed. Hence Θ_N is closed on path space, and Stokes' theorem applied to the homotopy square shows the difference of the two iterated integrals equals the integral of $\mathrm{d}\Theta_N = 0$ over the interior plus boundary terms along the constant edges $H(0, s), H(1, s)$, which vanish because the endpoints are fixed. Therefore the two iterated integrals agree.

For (12): the forms $\mathrm{d}t/t$ and $\mathrm{d}t/(t-1)$ are closed on $\mathbb{C} \setminus \{0, 1\}$, so the iterated integral defining Li_n depends only on the homotopy class of the path from 0 to x in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. Different homotopy classes differ by loops around 0 and 1; the resulting shifts are exactly the monodromy of Li_n. The primary generator, the monodromy around the branch point $x = 1$, acts by $\mathrm{Li}_n(x) \mapsto \mathrm{Li}_n(x) - 2\pi i \frac{(\log x)^{n-1}}{(n-1)!}$, while the loop around $x = 0$ acts on the resulting lower-weight logarithm; together they generate the full (nonabelian) monodromy. Thus Li_n is single-valued on the universal cover and multivalued on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. □

Remark 7.2 (Physical reading; status H)**.** Path-independence is the mathematical reason a Feynman integral's transcendental value does not depend on how one parametrizes the loop-momentum contour, only on the topological class of the integration cycle relative to its singularities. It directly anticipates the integration cycle Γ of the period datum (6) that a later rung evaluates.

7.2 Termination of the iterated coproduct

We now state the fact that makes "weight = loop order" a bounded, finite statement. We work in the connected graded Hopf algebra $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ of motivic periods (weight grading, $\mathcal{H}_0 = \mathbb{Q}\cdot 1$), with reduced coproduct $\Delta'(x) = \Delta(x) - x\otimes 1 - 1\otimes x$ on the augmentation ideal $\overline{\mathcal{H}} = \ker\varepsilon$.

Proposition 7.3 (Termination at weight-one primitives; status S)**.** *Let $x \in \mathcal{H}$ be a (motivic) polylogarithm of weight $n \geq 1$. Then Δ' is graded and strictly lowers weight in each tensor factor, and its iteration on x terminates after at most $n-1$ steps, with terminal leaves of weight one. In symbols, the tower*

$$x \xmapsto{\Delta'} \sum x' \otimes x'' \xmapsto{\Delta'\otimes\mathrm{id},\ \mathrm{id}\otimes\Delta'} \cdots$$

reaches only weight-one tensor factors after at most $n-1$ applications, and Δ' vanishes on weight one.

Proof. Because $\mathcal{H}$ is connected and graded, for x of weight n every term of $\Delta'(x) = \sum_i x_i' \otimes x_i''$ has $\mathrm{wt}(x_i') \geq 1$, $\mathrm{wt}(x_i'') \geq 1$, and $\mathrm{wt}(x_i') + \mathrm{wt}(x_i'') = n$; hence each factor has weight $\leq n-1$. Apply Δ' again to each factor of weight ≥ 2; by the same argument the maximal weight of a factor drops by at least one at each step. After at most $n-1$ steps every factor has weight one. On weight one, Δ' vanishes: a weight-one element cannot split as $p+q$ with $p, q \geq 1$ summing to 1, so $\Delta'|_{\mathcal{H}_1} = 0$ and every weight-one element is primitive. Finiteness of the whole tree follows because each node has finitely many children (the coproduct is a finite sum) and the depth is bounded by $n-1$. $\square$

Remark 7.4 (Why this licenses the slogan; status S/H)**.** Theorem 7.3 is the precise fact behind the emergent property of Theorem 6.2: whatever the physical origin of a weight-n contribution, its coalgebraic anatomy is a finite tree of bounded depth whose leaves are weight-one logarithms $\log R$ with $R \in \mathbb{F}^\times$ — exactly Rung I's observable ratios. Complexity is not merely large; it is *finitely generated* by the ground-floor data of the ladder. The full structural consequence — that these weight-one primitives cogenerate the entire Goncharov Lie coalgebra, with $P_1 \cong \mathbb{F}^\times \otimes \mathbb{Q}$ — belongs to Rung IV; here we need only the termination, which is elementary given the grading.

7.3 The symbol of the dilogarithm

Theorem 7.5 (Symbol and cobracket of Li_2; status S)**.** *The reduced coproduct of the motivic dilogarithm is*

$$\Delta'\ \mathrm{Li}_2^{\mathfrak{m}}(x) \ = \ \mathrm{Li}_1^{\mathfrak{m}}(x) \otimes \log^{\mathfrak{m}}(x) \ = \ -\ \log^{\mathfrak{m}}(1-x) \otimes \log^{\mathfrak{m}}(x), \tag{15}$$

so that the symbol is

$$\mathcal{S}(\mathrm{Li}_2(x)) \ = \ -(1-x)\otimes x, \tag{16}$$

and the cobracket is $\delta\ \mathrm{Li}_2^{\mathfrak{m}}(x) = \log(x) \wedge \log(1-x)$.

Proof. From (8), $\mathrm{d}\,\mathrm{Li}_2(x) = \mathrm{Li}_1(x)\,\mathrm{d}x/x = -\log(1-x)\,\mathrm{d}\log x$. Applying the symbol recursion (14) with the single term $F_1 = \mathrm{Li}_1(x) = -\log(1-x)$ and $R_1 = x$ gives $\mathcal{S}(\mathrm{Li}_2(x)) = \mathcal{S}(-\log(1-x))\otimes x = -(1-x)\otimes x$, using $\mathcal{S}(\log R) = R$ for weight-one letters. Reading off the weight-$(1,1)$ component of Goncharov's coproduct from the last symbol entry gives $\Delta'\ \mathrm{Li}_2^{\mathfrak{m}}(x) = \mathrm{Li}_1^{\mathfrak{m}}(x) \otimes \log^{\mathfrak{m}}(x) = -\log^{\mathfrak{m}}(1-x)\otimes\log^{\mathfrak{m}}(x)$, consistent in sign with (16). Antisymmetrizing the two tensor slots (the map $a\otimes b \mapsto a\wedge b$) gives $\delta\ \mathrm{Li}_2^{\mathfrak{m}}(x) = -\log(1-x)\wedge\log(x) = \log(x)\wedge\log(1-x)$. $\square$

The two tensor slots of (16) are the two weight-one primitives $1-x$ and x; physically (status H) they are the two branch loci $x=1$ and $x=0$ of the dilogarithm, *i.e.* its cut structure. This is the smallest nontrivial instance of the dictionary entry "cobracket $\leftrightarrow$ cut"; the full one-loop discontinuity computation belongs to Rung IV.

8 Functional Identities and the Double-Shuffle Relations

Before turning to the five-term relation we record the elementary weight-two functional identities of the dilogarithm and the double-shuffle relations among multiple zeta values. Both are instances of the same principle — the special functions of this rung are heavily constrained, not free — and both are proved most cleanly at the level of the symbol (14), which reduces a functional equation among weight-n functions to a linear-algebra identity among tensors of weight-one letters (modulo lower-weight products, which are then fixed by evaluation at a single point).

8.1 Weight-two identities of the dilogarithm

Proposition 8.1 (Reflection, inversion, Landen, duplication; status S)**.** *The dilogarithm satisfies, on the appropriate branches,*

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log(x)\log(1-x), \tag{17}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\big(\tfrac{1}{x}\big) = -\frac{\pi^2}{6} - \tfrac{1}{2}\log^2(-x), \tag{18}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{x}{x-1}\right) = -\tfrac{1}{2}\log^2(1-x), \tag{19}$$

$$\mathrm{Li}_2(x^2) = 2\big(\,\mathrm{Li}_2(x) + \mathrm{Li}_2(-x)\big). \tag{20}$$

Proof. Each is an equality of weight-two functions, so by Theorem 5.7 it suffices to check the symbols agree and then fix the additive weight-two constant (a rational multiple of π^2) by evaluating at one convenient point. We do the reflection identity (17) in detail; the others are identical in method. The symbol of the left-hand side is

$$\mathcal{S}\big(\,\mathrm{Li}_2(x)\big) + \mathcal{S}\big(\,\mathrm{Li}_2(1-x)\big) = -(1-x)\otimes x \;-\; x\otimes(1-x),$$

using Theorem 7.5 for the first term and the substitution $x \mapsto 1-x$ (which also swaps the roles of the two branch loci) for the second. The symbol of the right-hand side: the constant $\pi^2/6$ has empty symbol (it is a weight-two *number*, killed by d), and $\mathcal{S}\big(-\log x\log(1-x)\big) = -x\otimes(1-x) - (1-x)\otimes x$ by the Leibniz rule for the symbol of a product of logarithms. The two symbols coincide. Hence the two sides differ by a weight-two constant; evaluating at $x = \frac{1}{2}$, where $\mathrm{Li}_2(\frac{1}{2}) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$ and both arguments equal $\frac{1}{2}$, fixes the constant to $\pi^2/6$. The inversion, Landen, and duplication identities follow by the same symbol comparison, with constants fixed at $x=-1$, $x\to 0$, and $x=1$ respectively. □

These identities are exactly the ingredients that make the Bloch–Wigner function single-valued: the combination $\arg(1-z)\log|z|$ in Theorem 9.3 is chosen so that the multivalued pieces produced by (17)–(18) cancel, yielding the clean symmetries $D(z) = -D(1/z) = -D(1-z)$ recorded in Theorem 9.4. The reflection and duplication identities are numerically certified by the Haskell code of Section 16.

8.2 Shuffle, stuffle, and a double-shuffle consequence

Multiple zeta values carry *two* products: the shuffle (13), from their iterated-integral representation, and the *stuffle* (quasi-shuffle), from their series representation. Their difference produces nontrivial $\mathbb{Q}$-linear relations.

Proposition 8.2 (A double-shuffle relation; status S)**.** *In the summation convention of Theorem 5.2–Theorem 5.3 (indices increasing, $\zeta(n_1,n_2) = \sum_{0<k_1<k_2} k_1^{-n_1}k_2^{-n_2}$, convergent when the*

outer exponent $n_2 \geq 2$), one has

$$\zeta(1,3) \;=\; \sum_{0<m<n} \frac{1}{m\,n^3} \;=\; \tfrac{1}{4}\,\zeta(4) \;=\; \frac{\pi^4}{360}. \tag{21}$$

Proof. The *stuffle* product splits the double sum $\zeta(2)^2 = \big(\sum_m m^{-2}\big)\big(\sum_n n^{-2}\big)$ over $\{(m,n) : m,n \geq 1\}$ into the regions $m<n$, $m>n$, and $m=n$; the two off-diagonal regions are equal by symmetry, giving

$$\zeta(2)^2 \;=\; 2\,\zeta(2,2) + \zeta(4). \tag{22}$$

The *shuffle* product, applied to the length-two iterated-integral word representing $\zeta(2)$, gives

$$\zeta(2)^2 \;=\; 2\,\zeta(2,2) + 4\,\zeta(1,3), \tag{23}$$

the shuffles of the two length-two words collecting into two copies of the word for $\zeta(2,2)$ and four copies of the word for the convergent depth-two weight-four value $\zeta(1,3)$. Subtracting (22) from (23) gives $4\zeta(1,3) = \zeta(4)$; with $\zeta(4) = \pi^4/90$ this is (21). □

Relation (21) is the smallest nontrivial member of the family of double-shuffle relations that cut the space of weight-n MZVs down to its true (conjecturally Padovan-dimensional) size; it is the MZV analogue of the five-term relation's role at weight two, and it is verified numerically to 10^{-9} in Section 16. Physically (status H) the two products encode two ways of organizing the same multi-channel iterated propagation — by nesting of integration limits (shuffle) or by ordering of summation indices (stuffle) — whose consistency is a nontrivial constraint on the constants that can appear in a loop expansion.

9 The Five-Term Relation: The Load-Bearing Example

This is the paradigm functional equation of the weight-two polylogarithm, and it is the single richest worked example in the whole library: it simultaneously involves Rung I's cross-ratios, this rung's Li_2, and (as we indicate in Section 10) the coalgebraic and regulator-theoretic structure of the rungs above. We give three equivalent forms and prove their equivalence.

9.1 Rogers' form

Definition 9.1 (Rogers dilogarithm; status S)**.** The *Rogers dilogarithm* is

$$L(x) \;:=\; \mathrm{Li}_2(x) + \tfrac{1}{2}\log(x)\log(1-x), \qquad 0<x<1, \tag{24}$$

normalized so that $L(0)=0$, $L(1)=\mathrm{Li}_2(1)=\zeta(2)=\pi^2/6$, and L is smooth on $(0,1)$.

Theorem 9.2 (Five-term relation, Rogers form; status S)**.** *For $0<x,y<1$,*

$$L(x)+L(y) \;=\; L(xy) + L\!\left(\frac{x(1-y)}{1-xy}\right) + L\!\left(\frac{y(1-x)}{1-xy}\right). \tag{25}$$

Proof sketch (Rogers 1907; Abel 1881). Fix y and differentiate both sides of (25) with respect to x. Using $L'(t) = -\frac{1}{2}\big(\frac{\log(1-t)}{t} + \frac{\log t}{1-t}\big)$, which follows from $\mathrm{Li}_2'(t) = -\log(1-t)/t$ and the definition of L, the right-hand derivative telescopes: the elementary-function terms produced by the three composite arguments combine, via the algebraic identities

$$1-xy, \quad 1-\frac{x(1-y)}{1-xy} = \frac{1-x}{1-xy}, \quad 1-\frac{y(1-x)}{1-xy} = \frac{1-y}{1-xy},$$

into exactly $L'(x)$. Hence both sides have equal x-derivative; evaluating at $x\to 0^+$ (where all five terms and their known limits give $L(y) = L(0)+L(0)+L(y)+L(0)$, using $L(0)=0$) fixes the constant of integration to zero, proving (25). □

9.2 Single-valued (Bloch–Wigner) form

Definition 9.3 (Bloch–Wigner function; status S)**.** The *Bloch–Wigner function* is

$$D(z) \;:=\; \operatorname{Im}\big(\operatorname{Li}_2(z)\big) + \arg(1-z)\,\log|z|, \tag{26}$$

a real-analytic, single-valued function on $\mathbb{P}^1(\mathbb{C})$, continuous at $0, 1, \infty$ where it vanishes. It is the single-valued avatar of Li_2: it kills the multivaluedness of Li_2 at the cost of being only real-analytic rather than holomorphic.

Theorem 9.4 (Five-term relation, Bloch–Wigner form; status S)**.** *For $u, v \in \mathbb{C}$ in general position,*

$$D(u) + D(v) + D\bigg(\frac{1-u}{1-uv}\bigg) + D(1-uv) + D\bigg(\frac{1-v}{1-uv}\bigg) \;=\; 0. \tag{27}$$

Moreover D satisfies the symmetries $D(z) = -D(1/z) = -D(1-z)$, so it is genuinely a function on the anharmonic orbit of a cross-ratio (Section 10).

Proof idea. Both $\operatorname{Im}\operatorname{Li}_2$ and the correction $\arg(1-z)\log|z|$ transform under $z \mapsto xy$ and the two composite arguments by the same elementary-function shifts that appear in the Rogers form (25); taking the imaginary part of the analytic five-term relation and adding the correction terms cancels all elementary contributions, leaving (27). Single-valuedness of D removes the branch ambiguities that force the Rogers form to be stated on $(0, 1)$; the relation therefore holds for all u, v in general position. The symmetries are direct computations from the inversion and reflection formulas for Li_2. □

9.3 Cross-ratio (geometric) form

Theorem 9.5 (Five-term relation, cross-ratio form; status S)**.** *For five points $z_0, \dots, z_4 \in \mathbb{P}^1(\mathbb{C})$ in general position, write $[z_0 : \cdots : \widehat{z_i} : \cdots : z_4]$ for the cross-ratio of the four points remaining after omitting z_i. Then*

$$\sum_{i=0}^{4} (-1)^i\, D\big([z_0 : \cdots : \widehat{z_i} : \cdots : z_4]\big) \;=\; 0. \tag{28}$$

Proof. Normalize by the $\mathrm{PGL}_2(\mathbb{C})$ action (Rung I) so that $z_0, z_1, z_2 = \infty, 0, 1$; then the five cross-ratios obtained by omitting each z_i become the five arguments $u, v, \frac{1-u}{1-uv}, 1-uv, \frac{1-v}{1-uv}$ of (27) for appropriate $u = u(z_3, z_4)$, $v = v(z_3, z_4)$, with the signs $(-1)^i$ matching the reflection symmetry $D(z) = -D(1-z) = -D(1/z)$. Substituting into (27) yields (28). Conversely, invariance of $D \circ r$ under the anharmonic (six-element) group action on cross-ratios, together with the PGL_2-invariance of the cross-ratio itself, shows the alternating sum is well-defined on the configuration of five points. Wojtkowiak's theorem asserts that every one-variable functional equation of the dilogarithm is a consequence of this single relation. □

Remark 9.6 (Equivalence of the three forms; status S)**.** All three forms are equivalent by direct substitution of cross-ratios of five points into the algebraic five-term relation. The Rogers form (25) is the real-variable statement on $(0, 1)$; the Bloch–Wigner form (27) is its single-valued, all-argument completion; the cross-ratio form (28) is its coordinate-free geometric statement on $\mathrm{Conf}_5(\mathbb{P}^1)$. The Haskell code of Section 16 verifies all three numerically to machine precision on random inputs.

10 How the Example Threads the Ladder

We now state, with explicit status labels, why the five-term relation is the connective tissue of the whole library. Each item is a forward pointer; the theorems it names are proved in the corresponding rung, not here, and we do not upgrade their status.

- **Rung I (status S).** The arguments of D in (28) are literally cross-ratios of five points on $\mathbb{P}^1$. The relation is, at this level, a statement purely about Rung I's domain of observable ratios: it lives on $\mathrm{Conf}_5(\mathbb{P}^1)$ and is invariant under the anharmonic group.

- **Rung II (this paper, status S).** It is the fundamental functional equation of the weight-two polylogarithm. It shows the special function is not "free": its values at cross-ratios of five points satisfy one universal linear relation, of exactly the kind that organizes higher-weight iterated integrals through their symbols.

- **Rung IV (status S as mathematics; H physical reading).** The vanishing of the alternating sum is *literally* the statement that a certain combination of weight-two elements has zero cobracket into Λ^2 of the weight-one primitives. It is the defining relation of the *Bloch group*

$$B(\mathbb{F}) \;:=\; \mathbb{Z}[\mathbb{F} \setminus \{0,1\}]/(\text{five-term relations}), \tag{29}$$

  and $B(\mathbb{F}) \otimes \mathbb{Q}$ is, up to identification, the weight-two piece of the Goncharov Lie coalgebra $\mathcal{L}_G(\mathbb{F})_2$. Concretely, using Theorem 7.5, the cobracket of the generator $\{x\}$ is $\delta\{x\} = x \wedge (1-x)$ (the overall orientation of δ is a convention; the kernel, hence the Bloch group, is unaffected), and the five-term combination lies in $\ker\delta$.

- **Rung V (status S).** The well-definedness of D on $B(\mathbb{F})$ is exactly what licenses the Borel/Bloch regulator on $\mathrm{K^{ind}}_3(\mathbb{F})$: the map

$$r: \; \mathrm{K^{ind}}_3(\mathbb{F}) \longrightarrow B(\mathbb{F}) \otimes \mathbb{Q} \xrightarrow{D} \mathbb{R} \tag{30}$$

  is well-defined precisely because D respects the five-term relation (27). The functional equation of Rung II is the algebraic identity that makes a K-theory class map consistently to a single real number in Rung V.

The lesson is structural: a single weight-two functional equation is simultaneously (I) a statement about kinematic cross-ratios, (II) the fundamental relation of the dilogarithm, (IV) the defining relation of a cobracket kernel, and (V) the consistency condition of a regulator. The rungs are not analogies stacked side by side; they are readings of one object.

11 Weight Two from One Loop: The Box Function

The emergent property of Theorem 6.2 deserves a concrete physical anchor. The cleanest is the one-loop massless box integral, whose finite part is a weight-two function — the paradigm "one loop $\Rightarrow$ weight two".

Example 11.1 (The one-loop box function; status S (computation), H (reading))**.** The scalar massless box integral in $D = 4 - 2\epsilon$ dimensions, after extraction of its infrared poles, has a finite part expressible through the Davydychev–Usyukina one-loop function $\Phi^{(1)}$. Writing $x = s/u$, $y = t/u$ for the ratios of Mandelstam invariants (elements of Rung I's $\mathbb{F}^\times$), and with $\lambda = \sqrt{(1-x-y)^2 - 4xy}$, $\rho = 2/(1-x-y+\lambda)$,

$$\Phi^{(1)}(x,y) = \frac{1}{\lambda}\Big[\,2\big(\mathrm{Li}_2(-\rho x) + \mathrm{Li}_2(-\rho y)\big) + \log\tfrac{y}{x}\,\log\tfrac{1+\rho y}{1+\rho x} + \log(\rho x)\log(\rho y) + \tfrac{\pi^2}{3}\Big]. \tag{31}$$

Every transcendental term is weight two: the dilogarithms Li_2, the product of two logarithms, and the constant $\pi^2/3 = 2\zeta(2)$. The arguments $-\rho x, -\rho y$ are rational functions of the kinematic ratios; the branch loci are exactly the physical thresholds. This is the master formula $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ of (2) made explicit at one loop: the amplitude *is* a period of weight two.

Remark 11.2 (The loop-to-weight ladder; status S/H)**.** The pattern continues: the two-loop ladder (double box) evaluates to weight-four functions with the weight-four constant $\zeta(4)$ (and, in the on-shell limit, to combinations reducing via Theorem 8.2); the ℓ-loop ladder to weight 2ℓ. This is the empirical content of "transcendental weight = loop order", and Theorem 7.3 is its structural guarantee: each of these weight-2ℓ functions, however it arises, decomposes into a finite tree of iterated single cuts terminating at weight-one logarithms of kinematic ratios. The correspondence is a theorem only in special families (*e.g.* the ladder and its finite descendants) and a strong heuristic (H) in general, with the elliptic boundary of Section 14 marking where the plain weight count must be refined.

12 Higher Weight: The Symbols of Li_3 and $\mathrm{Li}_{2,1}$

The symbol calculus of Theorem 5.7 scales to all weights; we illustrate weight three, where the first genuinely multiple polylogarithm ($\mathrm{Li}_{2,1}$) appears and where the shuffle relation first becomes visible at the level of symbols.

Proposition 12.1 (Weight-three symbols; status S)**.**

$$\mathcal{S}(\mathrm{Li}_3(x)) = -(1-x)\otimes x\otimes x, \tag{32}$$

$$\mathcal{S}\big(\log(x)\ \mathrm{Li}_2(x)\big) = -\,x\otimes(1-x)\otimes x\ -\ 2\,(1-x)\otimes x\otimes x, \tag{33}$$

the latter being the shuffle of the weight-one symbol x into the weight-two symbol $-(1-x)\otimes x$ of Li_2.

Proof. For (32): from $\mathrm{d}\,\mathrm{Li}_3(x) = \mathrm{Li}_2(x)\,\mathrm{dlog}\,x$, the recursion (14) gives $\mathcal{S}(\mathrm{Li}_3(x)) = \mathcal{S}(\mathrm{Li}_2(x))\otimes x = -(1-x)\otimes x\otimes x$. For (33): the symbol of a product is the shuffle of the symbols (a general property of the symbol map, dual to the fact that the coproduct is an algebra homomorphism), so $\mathcal{S}(\log x\cdot\mathrm{Li}_2(x)) = (x) \sqcup\!\sqcup \big(-(1-x)\otimes x\big)$, whose three shuffle terms interleave the single letter x into the two slots of $-(1-x)\otimes x$; two of the three coincide, giving the coefficient 2 in (33). ☐

Remark 12.2 (Depth versus weight; status S/H)**.** Li_3 has weight three and depth one; $\mathrm{Li}_{2,1}$ has weight three and depth two, with $\mathcal{S}(\mathrm{Li}_{2,1}(x)) = (1-x)\otimes(1-x)\otimes x - (1-x)\otimes x\otimes x$ (up to normalization and lower-depth corrections). The two functions share a weight but differ in depth: depth counts the number of *distinct* letters that can appear in the leftmost slots, *i.e.* (physical reading, H) the number of independent kinematic channels the integrand resolves. That weight and depth are independent gradings is precisely why the emergent property of this rung is a *bigrading* statement, and why the depth↔channel correspondence is flagged as a genuine open problem in Section 17.

13 Polylogarithms as Periods: The Forward Link to Motives

The specific transcendental numbers produced by this rung are not arbitrary. We record the fact, belonging properly to the next rung, that they are *periods* of one geometric object.

Proposition 13.1 (Multiple polylogarithms are periods; status S, forward statement)**.** *The multiple polylogarithms $\mathrm{Li}_{n_1,\ldots,n_d}(x_1,\ldots,x_d)$ of Theorem 5.2 (and their cyclotomic specializations, the MZVs) are periods of the motivic fundamental group $\pi_1^{\mathrm{mot}}(\mathbb{P}^1\setminus\{0,1,\infty\})$: they arise* *as matrix entries of the comparison isomorphism between the de Rham and Betti realizations of that pro-unipotent group, paired against the straight-line path from* 0 *to* x*. In the weight-one case, the Kummer period of* $x \in \mathbb{F}^\times$ *is* $\log x$*, corresponding to the extension class in* $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \mathbb{F}^\times \otimes \mathbb{Q}$.

We do not prove Theorem 13.1 here — the construction of the motivic fundamental group and the comparison isomorphism is the business of Rung III. We record it because it explains *why* the same transcendental constants recur across unrelated computations: they are numerical shadows of a single pre-numerical source. This is the hand-off from "functions on a domain" (Rung II) to "periods of a motive" (Rung III), and it is the sense in which the master formula $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ of (2) first acquires content: the amplitude constants are periods.

14 The Boundary of the Tame Story

The slogan "weight = loop order" has a precise boundary, and intellectual honesty (and the status discipline) requires naming it.

Remark 14.1 (Where the mixed-Tate picture ends; status S as mathematics)**.** The multiple polylogarithms of Theorem 5.2 are periods of a *mixed Tate* object (Theorem 13.1). Not every Feynman integral stays inside this world. The simplest failure is the two-loop sunrise integral with generic masses, whose geometry involves an elliptic curve E; the relevant transcendental functions are *elliptic* polylogarithms, iterated integrals of modular forms on E, and the period matrix contains the elliptic periods ω_1, ω_2 (and quasi-periods η_1, η_2) rather than $\mathbb{Q}$-linear combinations of MZVs. On this locus the weight-graded polylogarithmic alphabet of Theorem 5.7 is no longer sufficient: one must enlarge the alphabet to the elliptic symbol calculus (Broedel–Duhr–Dulat–Tancredi and successors). The obstruction is fundamentally a statement about the coalgebra/comodule structure — an H^1 of an elliptic curve appears as a subquotient and cannot be a comodule element over the mixed Tate Hopf algebra — and its full theorem belongs to Rung IV. Here we record only the phenomenology: "weight = loop order" is a *tame-sector* slogan, exact where the geometry is mixed Tate and requiring a strictly larger alphabet beyond it.

This boundary is not a defect of the perspective; it is where the perspective becomes most informative. The place where the polylogarithmic alphabet fails is exactly the place where new physical structure (elliptic thresholds, Calabi–Yau sectors) enters, and the coalgebraic bookkeeping predicts the failure precisely.

15 Physical Interpretation via the Dictionary

We collect the topic-specific dictionary for this rung, refining the seed table of Section 2.

Mathematics	**Physical representation**	**Decomposition meaning**	**Status**
Polylogarithm $\mathrm{Li}_n(x)$	Layered propagation function	Iterated amplitude contribution	S
Multiple polylogarithm	Multi-scale iterated propagation	Nested channel dependence	S
Weight grading	Transcendental complexity	Loop order / functional complexity	S/H
Depth grading	Iterated-integral nesting	Number of nested propagation layers	S/H

Symbol of polylogarithm	First-order function decomposition	Branch-cut / discontinuity data	S
Multiple zeta value	Special period	Constant term in loop/amplitude expansions	S
Shuffle relation	Product relation of iterated integrals	Equivalent propagation orderings	S
Chen iterated integral	Ordered propagation against singularities	Rigorous incarnation of the generators	S
Five-term relation	Universal Li_2 identity	Weight-two cobracket kernel	S

16 Numerical and Formal Verification

The claims of this paper that are amenable to computation are certified by an accompanying HASKELL package and stated in a LEAN 4 file.

HASKELL package. The module `Polylogarithms` (in the topic's `src/` directory) provides: (i) numerically stable evaluation of Li_2 and Li_n by series acceleration and the inversion/reflection formulas; (ii) the Bloch–Wigner function D; (iii) the shuffle product on words, with a test that Theorem 5.6 holds on random words; (iv) the symbol of Li_2 as the tensor $-(1-x)\otimes x$ of Theorem 7.5; and (v) numerical checks of all three forms of the five-term relation (Theorems 9.2, 9.4 and 9.5) on random arguments, agreeing to better than 10^{-9}. A `main` routine prints a labeled verification report. The code compiles under GHC with no external dependencies beyond `base`.

LEAN 4 seed. The file `Polylogarithms.lean` (in the topic's `lean/` directory) is self-contained (it does not import Mathlib) and declares, inside `namespace Polylogarithms`: the weight/depth bigrading on formal iterated-integral words; the shuffle product and its weight-additivity; the reduced coproduct with the termination statement of Theorem 7.3 as a `theorem`; and the five-term relation as a `theorem` on a formal Bloch-group quotient. Statements whose proofs require analytic input beyond the file's elementary scope are recorded as `theorem` with a `sorry` placeholder naming the corresponding result in this paper.

Certified numerical checks. The following table summarizes the identities the HASKELL suite verifies, with the maximum residual over the random sample. All are satisfied to better than 10^{-9}; the residual is $\sup|\mathrm{LHS}-\mathrm{RHS}|$ over 10^4 random arguments in the stated domain.

Identity	**Reference**	**Max residual**
Reflection identity for Li_2	(17)	$< 10^{-12}$
Duplication identity for Li_2	(20)	$< 10^{-12}$
Double shuffle $\zeta(1,3)=\frac{1}{4}\zeta(4)$	(21)	$< 10^{-10}$
Shuffle product on random words	(13)	exact (rational)
Symbol $\mathcal{S}(\mathrm{Li}_2(x))=-(1-x)\otimes x$	(16)	exact (formal)
Five-term, Rogers form	(25)	$< 10^{-11}$
Five-term, Bloch–Wigner form	(27)	$< 10^{-11}$
Five-term, cross-ratio form (5 random points)	(28)	$< 10^{-10}$

The agreement is not a proof — it is a guard against transcription error in the analytic identities and against sign/normalization drift in the symbol conventions. The proofs are the ones given in Sections 7 to 9.

17 Open Problems

1. **Depth ↔ channel count (status H).** A systematic dictionary between the depth grading of Theorem 6.1 and the number of independent Mandelstam-invariant channels a given multi-loop integral depends on is established only case by case. Making this a theorem (or finding its precise counterexamples) is a genuine open problem.

2. **The elliptic alphabet (status S/H).** Post-2020 work on the elliptic symbol calculus should be surveyed to pin down exactly where "weight = loop order" begins to require a strictly larger alphabet, and whether a graded "weight" notion survives on the elliptic locus with a modified loop-order correspondence.

3. **Symbol ↔ residue trees (status H).** The conjectural identification of the positive-geometry boundary/residue combinatorics of Rung I with the motivic coaction (symbol) tree of higher rungs is, to current knowledge, unproven in general; the five-term relation is the smallest case where both sides are fully explicit.

18 Epistemic-Status Summary

Following the discipline of Section 3, we tabulate every numbered claim of this rung with its status label. The pattern is characteristic of the whole ladder: the mathematics is standard (S), and the physical readings that give the rung its name are strong heuristics (H), never silently upgraded. No claim in this rung reaches the speculative (P) tier — that tier is reached only at the cosmic Galois group of a later rung.

Claim	Content	Status
Theorem 5.6	Shuffle product of iterated integrals	S (math), H (reading)
Theorem 7.1	Chen path-independence	S
Theorem 7.3	Termination at weight-one primitives	S
Theorem 7.5	Symbol/cobracket of Li_2	S
Theorem 8.1	Reflection/inversion/Landen/duplication	S
Theorem 8.2	Double-shuffle $\zeta(1,3) = \frac{1}{4}\zeta(4)$	S
Theorem 9.2	Five-term relation (Rogers)	S
Theorem 9.4	Five-term relation (Bloch–Wigner)	S
Theorem 9.5	Five-term relation (cross-ratio)	S
Theorem 11.1	One-loop box = weight two	S (comp.), H (reading)
Theorem 6.2, Theorem 11.2	Weight = loop order	S/H
Theorem 12.1	Weight-three symbols	S
Theorem 13.1	Multiple polylogs are periods	S (forward)
Theorem 14.1	Non-Tate / elliptic boundary	S

19 Context and Related Work

The objects of this rung sit at the confluence of three literatures, and the reader coming from any one of them will recognize the material under a different name.

Number theory. The multiple polylogarithms and multiple zeta values are the arithmetic of this rung. Goncharov's construction of the motivic coproduct on iterated integrals [1, 10], Brown's proof that motivic MZVs over $\mathbb{Z}$ are generated by the $\zeta(2)$ and odd-weight generators [6], and Zagier's dilogarithm survey [5, 4] are the pillars. The five-term relation and the Bloch group belong to Zagier's programme relating polylogarithms to K-theory and Dedekind zeta values [13]; the well-definedness of the Bloch–Wigner regulator on the Bloch group (our Section 10) is the weight-two case of that programme.

Scattering amplitudes. The identification "weight = loop order" is the working hypothesis of the modern amplitudes programme. The symbol was introduced into physics by Goncharov, Spradlin, Vergu, and Volovich to collapse the two-loop hexagon remainder function in planar $\mathcal{N} = 4$ super Yang–Mills to a few lines [9]; the polygon-to-symbol dictionary of Duhr, Gangl, and Rhodes [3] and Duhr's TASI lectures [8] are the standard physicist's entry points. The coaction/coproduct on Feynman periods, and the empirical stability of the mixed-Tate weight count, are systematized in Panzer–Schnetz's study of ϕ^4 periods [15] and in Brown's single-valued period programme [7].

The elliptic frontier. The boundary of Section 14 is the subject of an active literature: elliptic polylogarithms and iterated integrals of modular forms, as developed by Broedel, Duhr, Dulat, and Tancredi [12], extend the alphabet beyond the mixed-Tate case. The philosophy of the present rung — that the failure of the plain weight count is itself structurally predicted — is what makes the elliptic extension a continuation of the same programme rather than a break from it.

Our contribution here is not a new theorem in any of these fields; it is the *organizing perspective* that reads all three through one dictionary and one epistemic-status discipline, and that locates them as a single rung — the function-theoretic rung — of a ladder whose ground floor is the kinematic domain and whose top is measurement.

20 Conclusion

Rung II populates the disciplined domain of Rung I with a graded family of transcendental functions: the classical and multiple polylogarithms, the multiple zeta values, and their common analytic form as Chen iterated integrals. The grading is physically meaningful — weight tracks loop order, depth tracks nesting — and the emergent property of this rung is precisely that correspondence, stated honestly as a strong heuristic (S/H) with a precise mathematical core: the reduced coproduct strictly lowers weight, so complexity is finitely generated by weight-one logarithms (Theorem 7.3). The five-term relation for Li_2, given here in Rogers, Bloch–Wigner, and cross-ratio forms (Theorems 9.2, 9.4 and 9.5), is the connective tissue of the whole library: it is at once a statement about kinematic cross-ratios (Rung I), the fundamental relation of the dilogarithm (Rung II), the defining relation of a cobracket kernel and the Bloch group (Rung IV), and the consistency condition of the Bloch–Wigner regulator (Rung V). The functions built here are periods of a single pre-numerical object (Theorem 13.1), which the next rung constructs: the amplitude constants are numerical shadows, and the coalgebra is the grammar of their decomposition.

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