Motives as the Pre-Numerical Object: Betti, de Rham, and Hodge Realizations of Physical Information

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Abstract

A recurring puzzle in the mathematics of scattering amplitudes and Feynman integrals is that the *same* transcendental constants — π, $\zeta(3)$, logarithms, multiple polylogarithm values — reappear across physically unrelated computations. This paper advances a structural explanation from the theory of motives: those constants are not independent numerical facts but *periods*, the numerical shadows of a single, deeper, pre-numerical object. Adopting the governing perspective of a standalone modular library — that a Goncharov-style Lie coalgebra is a bookkeeping system for how physical observables decompose into irreducible informational constituents — we treat the motive $\mathrm{Mot}(X)$ of a variety as the invariant object from which every cohomological reading, and hence every measured number, is *realized*. We develop the four classical realization functors (Betti, de Rham, Hodge, and étale), the comparison isomorphism relating them, and the interpretation of periods as its matrix entries. Two rigorous anchors are given: the existence of a Tannakian category of mixed realizations with functorial Betti/de Rham/étale images and their comparisons (after Huber), and the unconditional construction of the category $\mathrm{MTM}(k)$ of mixed Tate motives over a number field (after Levine, Deligne–Goncharov, and Bloch–Kriz). We work the Kummer motive in detail, exhibiting $\log x$ as the period of a nonsplit extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ classified by $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q}$, and we identify multiple polylogarithms as periods of the motivic fundamental group $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. Throughout we apply a verbatim mathematics-to-physics dictionary and a three-tier epistemic status system (Standard / Heuristic / Philosophical), so that the rigorous mathematical content is never conflated with its proposed physical reading. The emergent property named by this rung of the ladder is precisely *the pre-numerical object behind measurement*: the recognition that a measured amplitude is $\mathrm{per}(\mathcal{A}^{\mathfrak{m}})$, the period of a motivic object $\mathcal{A}^{\mathfrak{m}}$, and that the object — not the number — is the primary carrier of physical information.

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1 Introduction

1.1 The recurrence puzzle

Perturbative quantum field theory produces numbers. A Feynman integral, once regularized and renormalized, evaluates to a definite real (or complex) constant; a scattering amplitude at a fixed kinematic point is likewise a number. What is striking — and what motivates this paper — is that the numbers are not arbitrary. Across computations with no obvious common structure, the same constants recur: powers of π, the Riemann zeta values $\zeta(3)$, $\zeta(5)$, …, ordinary logarithms of kinematic ratios, and the values of classical and multiple polylogarithms. The constants form a small, structured, arithmetically special set. Why should this be so?

The thesis of this paper is that the recurrence is not a coincidence of computation but a shadow of geometry. The transcendental constants of perturbative physics are *periods*: numbers obtained by integrating an algebraic differential form over a cycle in an algebraic variety. And periods, in turn, are the numerical realizations of a single pre-numerical object attached to that variety — its *motive*. Two computations produce the same constant because they are pairing (possibly different) forms and cycles that descend from the same underlying motivic structure. The number is a projection; the motive is what is projected.

1.2 The governing perspective

This paper is the third rung of a modular research library whose organizing sentence is:

> A Goncharov-style Lie coalgebra is not merely a formal algebraic object; it is a *bookkeeping system* for how physical quantities break apart into irreducible informational constituents.

A physical observable is treated as the numerical realization of a deeper, structured, pre-numerical object, and the question "what is this observable made of?" is to be answered not by further computation but by a coalgebraic decomposition law. The full library builds this apparatus rung by rung. Rung I supplies the disciplined *domain* of observable values — fields, multiplicative groups, cross-ratios, moduli of marked points. Rung II supplies the *functions* on that domain: classical and multiple polylogarithms, Chen's iterated integrals, and the weight/depth grading that tracks loop order. The present Rung III supplies the *object* whose numerical shadows those functions' special values are. Rung IV (the core) supplies the *decomposition law* — the Hopf-algebraic coproduct and the Goncharov Lie coalgebra's cobracket. Rung V supplies the *evaluation* — the regulator and period pairing that turn the object back into a measurable number. A synthesis rung recomposes the ladder and identifies measurement as its closure.

This is a *modular* framework, not a unified one. Each rung is a standalone, defensible development; each composes with its neighbour to add one emergent structural property the level below cannot express. The emergent property of the present rung is stated plainly in its title: the motive is the pre-numerical object behind measurement.

1.3 What this paper contributes

We do not claim new theorems of pure motive theory. Our contribution is threefold and expository-structural in character.

1. We assemble, in a single self-contained treatment aimed at a mathematically literate physics readership, the realization picture of motives: the four classical realization functors, the comparison isomorphism, and the period interpretation, anchored to two genuine existence theorems (Theorems 7.1 and 7.2).

2. We work the two load-bearing examples of the whole library from the motivic side: the Kummer motive, whose period is $\log x$ and which is classified by $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q}$ (Theorem 7.4), and the motivic fundamental group of the thrice-punctured line, whose periods are the multiple polylogarithms (Theorem 7.7).

3. We articulate — with explicit epistemic status labels throughout — the proposed physical semantics of these objects, being scrupulous never to conflate the "functional-essence" reading of a motive (our heuristic) with Grothendieck's actual definition (standard mathematics). This discipline is the subject of Sections 2 and 3.

1.4 Outline

Section 2 reproduces the verbatim mathematics-to-physics dictionary. Section 3 sets out the Standard/Heuristic/Philosophical status system and its composition rule. Section 4 fixes the master formulas and notation shared across the library. Section 5 develops motives, Tannakian categories, and the realization functors, and states the boxed caution on the functional-essence reading. Section 6 defines the realization chain and the period interpretation of the comparison isomorphism. Section 7 states and proves (or cites with proof sketch) the core results, including the two worked examples. Section 8 draws out the emergent property — the pre-numerical object behind measurement — and its physical reading. Section 9 records the Non-Tate boundary of the tame story. Section 10 places the rung in the ladder. Section 11 discusses limitations and tabulates the status statistics of the paper. Section 12 concludes.

2 The Mathematics $\to$ Physics Dictionary

Every paper of this library opens with the same translation table, reproduced verbatim. It is the seed dictionary from which the topic-specific refinements (Sections 5 and 7) descend; the refinements extend it, they do not replace it.

Mathematics	Physical representation
Field F	Kinematic/coordinate domain of possible values
Element of $F^\times$	Ratio, cross-ratio, scale, observable parameter
Polylogarithm $\mathrm{Li}_n(x)$	Iterated propagation / layered amplitude contribution
Period	Measurable physical number obtained from geometry
Motive	Pre-numerical informational object behind the measurement
Lie algebra	Infinitesimal symmetry / generator structure
Lie coalgebra	Decomposition law of observables
Cobracket δ	Factorization channel, cut, boundary, or information split
Weight grading	Complexity depth / loop order / transcendental depth
Primitive element	Irreducible informational unit
Coproduct	Full hierarchical decomposition
Regulator map	Passage from motivic object to physical real number
Hodge/de Rham realization	Analytic form seen by calculation
Betti realization	Topological/path/integration-cycle form
Period pairing	Physical observable as pairing of form and cycle

Central slogans that recur across all papers of the library:

> observable = realization(motivic information)
> coalgebra = grammar of physical decomposition
> Physics observes numerical shadows of deeper motivic structures.
> Physical reality is the realization layer of structured mathematical information.

3 The Epistemic Status System (S / H / P)

Because this library proposes a semantic bridge between rigorous mathematics and physical interpretation, it is essential to mark, at every step, how load-bearing each claim is. We tag every dictionary entry and every mathematical statement with one of three status labels. The labels are part of the formalism, not decoration.

Label	Meaning	Canonical example
S	Standard use in mathematics or mathematical physics	The comparison isomorphism between de Rham and Betti cohomology
H	Strong heuristic representation-dictionary entry	Motive as "functional essence"; period as "measured number"
P	Speculative philosophical ontology	Physical reality as motivic-categorical realization

Composite labels **S/H** and **H/P** occur where an entry is standard as pure mathematics but heuristic or speculative in its *physical* reading. We preserve composite labels wherever they occur.

Remark 3.1 (Status monotonicity). If two representation entries or theorems compose, the status of the composite is the minimum under the order $\mathbf{S} < \mathbf{H} < \mathbf{P}$: a chain of translations is only as reliable as its weakest link. A **P**-labeled step cannot be upgraded to **S** merely by composing it with otherwise-rigorous results. This rule is applied when we tabulate the paper's status statistics in Section 11.

4 Master Formulas and Notation

The following notation is fixed across the entire library and used without further comment.

$$\text{observable} = \mathrm{Real}(\text{abstract structure}) \quad \text{compressed master slogan} \tag{1}$$

$$\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}}) \quad \text{amplitude = period of motivic amplitude object} \tag{2}$$

$$\Delta(\mathcal{A}^{\mathfrak{m}}) = \textstyle\sum_i \mathcal{A}^{\mathfrak{m}}_{i,1} \otimes \mathcal{A}^{\mathfrak{m}}_{i,2} \quad \text{coproduct / full hierarchical decomposition} \tag{3}$$

$$\delta : \mathcal{L} \to \bigwedge^2 \mathcal{L} \quad \text{cobracket / primitive antisymmetric split} \tag{4}$$

$$\mathrm{per}(\Pi) = \textstyle\int_\Gamma \omega, \quad \Pi = (X, D, \omega, \Gamma) \quad \text{period pairing: form paired with cycle} \tag{5}$$

Reading guide, fixed for the whole project:

- F is a field — typically a number field, or a field of kinematic invariants and rational functions of momenta; $F^\times$ its multiplicative group.
- X is a variety, $D \subset X$ a boundary divisor, ω an algebraic differential form on $X \setminus D$, and Γ a relative homology cycle of complementary dimension.
- $\mathcal{A}^{\mathfrak{m}}$ is a *motivic amplitude object*: a pre-numerical element of a Hopf algebra $\mathcal{H}$ of motivic periods; per is the period map sending it to the physical number $\mathcal{A}$.
- $\mathcal{L} = \bigoplus_{n \geq 1} \mathcal{L}_n$ is the (Goncharov) Lie coalgebra of primitives, graded by weight n; its construction and cobracket δ belong to Rung IV and are used here only as forward pointers.

- $\mathbb{Q}(n)$ denotes the n-th Tate twist, the pure motive with de Rham realization $\mathbb{C}$, Betti realization $(2\pi i)^n\mathbb{Q}$, and period $(2\pi i)^n$.

The present rung is responsible for the object $\mathcal{A}^{\mathfrak{m}}$ appearing in (2): for explaining what kind of thing a "motivic amplitude object" is, why the specific numbers of Rung II are its periods, and in what precise sense it is prior to the number $\mathcal{A}$.

5 Mathematical Framework: Motives and Their Realizations

5.1 Grothendieck's motives, informally

Grothendieck introduced motives to explain a structural regularity of algebraic geometry: the various cohomology theories of a smooth projective variety X — singular (Betti) cohomology of $X(\mathbb{C})$, algebraic de Rham cohomology, ℓ-adic étale cohomology for each prime ℓ, crystalline cohomology in positive characteristic — are not independent. They share Betti numbers, they carry compatible cup products, they satisfy the same formal properties (a *Weil cohomology theory*). Grothendieck's proposal was that each is a different *realization* of one universal object, the motive $\mathrm{Mot}(X)$, which sits above them all and through which every reasonable cohomology theory factors.

Definition 5.1 (Motive, informal; status **S**)**.** Grothendieck introduced *pure* (Chow) motives for smooth projective varieties as the universal cohomological invariant through which every Weil cohomology theory factors. The abelian category $\mathrm{MM}(k)$ of *mixed* motives over a base field $k \subset \mathbb{C}$ of characteristic zero — accommodating arbitrary varieties and carrying a functorial weight filtration — was conjectured and developed later by Beilinson, Deligne, and others. We write $\mathrm{Mot}(X)$ for the motive of X in this extended sense; it is designed so that every reasonable cohomology theory factors through a realization functor out of $\mathrm{MM}(k)$.

We emphasize at the outset the exact mathematical content of the word "motive," because the physical reading we will attach to it (Theorem 6.1) is easy to misconstrue.

Caution (to be kept in mind throughout). *Mathematically*, a motive is **not** "the functions of" an object, and it is not the ring of functions $\mathcal{O}(X)$ or any of its variants. A motive is a universal object unifying the different cohomology theories of X; it is the source of X's cohomological realizations, not its algebra of functions. The reading "functional essence" introduced in Theorem 6.1 below is a *proposed semantic layer* of this project (status **H**), adopted to align motives with the pipeline slogan observable = Real(abstract structure). This distinction is the origin of the entire S/H/P status discipline and must never be conflated with Grothendieck's standard mathematical definition.

5.2 Tannakian categories and Tate twists

The realization picture is cleanest in the Tannakian framework.

Definition 5.2 (Tannakian category; status **S**)**.** A neutral Tannakian category over $\mathbb{Q}$ is a $\mathbb{Q}$-linear abelian rigid tensor category $\mathcal{T}$ with a fibre functor $\omega : \mathcal{T} \to \mathrm{Vec}_{\mathbb{Q}}$ (an exact, faithful tensor functor to finite-dimensional $\mathbb{Q}$-vector spaces). Tannakian duality then gives a pro-algebraic group $G = \mathrm{Aut}^{\otimes}(\omega)$, the *Tannaka group*, such that $\mathcal{T}$ is equivalent to the category of finite-dimensional $\mathbb{Q}$-representations of G.

For motives, the relevant categories are $\mathrm{MM}(k)$ (mixed motives over k) and its subcategory $\mathrm{MTM}(k)$ of *mixed Tate* motives, built from the Tate twists $\mathbb{Q}(n)$ by iterated extension.

Definition 5.3 (Tate twist; status **S**)**.** The Tate motive $\mathbb{Q}(1)$ is the dual of $H^2(\mathbb{P}^1)$ (equivalently $H_2(\mathbb{P}^1)$); it is pure of *motivic weight* -2 and Hodge type $(-1,-1)$, with one-dimensional de Rham and Betti realizations. Its n-th tensor power is $\mathbb{Q}(n)$, of motivic weight $-2n$ and with period $(2\pi i)^n$. A mixed Tate motive is an object of $\mathrm{MM}(k)$ all of whose subquotients are isomorphic to Tate twists; $\mathrm{MTM}(k)$ is the full subcategory they generate. The simple objects of $\mathrm{MTM}(k)$ are exactly the $\mathbb{Q}(n)$, $n \in \mathbb{Z}$.

Remark 5.4 (Two gradings, do not conflate; status **S**)*.* Two distinct gradings appear in this library and must be kept apart. The *motivic (Hodge) weight* of $\mathbb{Q}(n)$ is $-2n$, as just defined. The *transcendental weight* used pervasively in Rungs II and IV is the positive grading with log in weight 1, Li_2 in weight 2, and in general Li_n in weight n; it tracks loop order and functional complexity. When we speak of "weight-one primitives $F^{\times} \otimes \mathbb{Q}$" we mean the transcendental grading; when we assign a motive a negative weight we mean the motivic grading.

5.3 The four realization functors

Definition 5.5 (Realization functor; status **S**)**.** A realization is an exact $\otimes$-functor $\mathrm{Real}_{\alpha} : \mathrm{MM}(k) \to \mathcal{T}_{\alpha}$ to a Tannakian target category $\mathcal{T}_{\alpha}$. The four classical realizations are:

- *Betti*, $\mathrm{Real}_{\mathrm{B}}$: singular cohomology $H^{\bullet}(X(\mathbb{C}), \mathbb{Q})$ of the complex points, valued in $\mathbb{Q}$-vector spaces equipped with a weight and (via comparison) a Hodge filtration — the *topological/path/cycle* side.
- *de Rham*, $\mathrm{Real}_{\mathrm{dR}}$: algebraic de Rham cohomology $H^{\bullet}_{\mathrm{dR}}(X/k)$, valued in filtered k-vector spaces — the *analytic/form/integrand* side.
- *Hodge*, $\mathrm{Real}_{\mathrm{Hdg}}$: the mixed Hodge structure on $H^{\bullet}(X(\mathbb{C}))$, valued in the Tannakian category of mixed Hodge structures.
- *Étale*, $\mathrm{Real}_{\text{ét},\ell}$: ℓ-adic cohomology $H^{\bullet}_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_{\ell})$ with its $\mathrm{Gal}(\bar{k}/k)$-action, valued in ℓ-adic Galois representations — the *arithmetic/discrete* side.

Definition 5.6 (Comparison isomorphism; status **S**)**.** For a smooth variety X over $k \subset \mathbb{C}$, the Betti and de Rham realizations of $\mathrm{Mot}(X)$ become isomorphic after extension of scalars to $\mathbb{C}$:

$$\mathrm{comp} : \ \mathrm{Real}_{\mathrm{dR}}(M) \otimes_k \mathbb{C} \ \xrightarrow{\ \sim\ } \ \mathrm{Real}_{\mathrm{B}}(M) \otimes_{\mathbb{Q}} \mathbb{C}, \tag{6}$$

natural in the motive M and compatible with tensor products and the weight and Hodge filtrations. Concretely, comp is the integration pairing: a de Rham class (an algebraic form) is paired against a Betti class (a topological cycle) by integration.

Definition 5.7 (Period; status **S**)**.** Fix a k-rational basis of $\mathrm{Real}_{\mathrm{dR}}(M)$ and a $\mathbb{Q}$-rational basis of $\mathrm{Real}_{\mathrm{B}}(M)$. The matrix of the comparison isomorphism (6) in these bases is the *period matrix* of M; its entries are the *periods* of M. Each entry is a number of the form $\int_\Gamma \omega$ for ω a de Rham basis form and Γ a Betti basis cycle.

Periods are, by Theorem 5.7, exactly the numerical content of the comparison isomorphism. The Kontsevich–Zagier period ring $\mathcal{P}$ [1] is the $\mathbb{Q}$-algebra generated by all such numbers as M ranges over motives of varieties over $\overline{\mathbb{Q}}$; it contains $\overline{\mathbb{Q}}$, $\log 2$, π, $\zeta(3)$, and the amplitude constants of perturbative QFT.

6 The Realization Chain and the Period Interpretation

6.1 The functional-essence reading

We now attach the physical semantics. It is a proposed reading, status **H**, and Theorem 5.1's caution box governs it.

Definition 6.1 (Functional-essence interpretation; status **H**)**.** Write $\mathrm{FE}(X) := \mathrm{Mot}(X)$, read as "the invariant, pre-numerical structure from which the various functional readings, cohomological realizations, periods, and numerical shadows of X are realized." Under this reading the motive is the *object* of physical information, and each realization functor is a particular *channel* through which that information is presented for calculation or measurement.

Definition 6.2 (Realization chain; status **S** as mathematics, **H** in its physical reading)**.** The realization chain is the composite

$$X \xmapsto{\mathrm{Mot}} \mathrm{Mot}(X) \xmapsto{\mathrm{Real}_\alpha} \mathrm{Real}_\alpha(\mathrm{Mot}(X)) \xmapsto{\mathrm{per}} \text{physical observable.} \tag{7}$$

Mathematically each arrow is a functor or a pairing; physically (status **H**), the chain reads as "geometry → pre-numerical object → channel-specific presentation → measured number."

The commuting square that makes "two calculations, one number" precise is:

$$\begin{tikzcd}
\mathrm{Real}_{\mathrm{dR}}(M) \otimes_k \mathbb{C} \arrow[r, "\mathrm{comp}", "\sim"'] \arrow[d, "\text{choose } \omega"'] & \mathrm{Real}_{\mathrm{B}}(M) \otimes_{\mathbb{Q}} \mathbb{C} \arrow[d, "\text{choose } \Gamma"] \\
\text{form side} \arrow[r, "\int_\Gamma \cdot"'] & \text{number } \ \mathrm{per}(\Pi) \in \mathbb{C}
\end{tikzcd}$$

Reading the square: whether one starts from the analytic (de Rham) side by choosing a form ω, or from the topological (Betti) side by choosing a cycle Γ, the pairing $\int_\Gamma \omega$ produces the *same* complex number. This is the motivic content of the physical statement that two different computational routes to an amplitude must agree.

6.2 The sculpture and its shadows

A useful and honest picture: the motive is a sculpture, and the realizations are its shadows cast on different walls. The Betti shadow records how cycles wind; the de Rham shadow records how forms integrate; the étale shadow records how Galois acts. The shadows look different, but they are functorial images of one object, not merely analogous descriptions — this is exactly what Theorem 7.1 below makes rigorous. The period is the measurement that reconciles two of the shadows: it is the entry of the dictionary comp translating de Rham coordinates into Betti coordinates.

$$\begin{tikzcd}
 & \mathrm{Mot}(X) \arrow[dl, "\mathrm{Real}_{\mathrm{B}}"'] \arrow[d, "\mathrm{Real}_{\mathrm{dR}}"] \arrow[dr, "\mathrm{Real}_{\text{ét}}"] & \\
\text{cycles/paths} & \text{forms/integrands} & \text{Galois data} \\
 & \mathrm{per}(\Pi) = \int_\Gamma \omega \arrow[u, dashed] &
\end{tikzcd}$$

7 Core Results

We now state the two structural existence theorems that license the whole picture, and then work the two examples that anchor the library.

7.1 Existence of comparison realizations

Theorem 7.1 (Existence of mixed realizations; status **S**, after Huber)**.** *There is a Tannakian category* $\mathrm{MR}(k)$ *of mixed realizations and a functor*

$$\mathrm{Real} : \ \mathrm{DM}_{\mathrm{gm}}(k) \ \longrightarrow \ D^b(\mathrm{MR}(k))$$

from Voevodsky's triangulated category of geometric motives, inducing on cohomology the classical Betti, de Rham, and ℓ-adic étale realizations together with their comparison isomorphisms.

Discussion and reference. This is the content of Huber's construction of the realization functor for Voevodsky's motives [3], with its published corrigendum; see also André's monograph [2] for the surrounding theory of motives and periods. The theorem is what upgrades the informal "sculpture and shadows" picture of Section 6 to a mathematical fact: the shadows (Betti/de Rham/étale) are honestly *functorial* images of one object in a triangulated category of motives, and the comparison isomorphisms between them are morphisms in $\mathrm{MR}(k)$, not merely numerical coincidences. We do not reproduce the construction; the point for this paper is that the object at the top of (7) exists and its realizations are functorial. □

7.2 Unconditional mixed Tate motives

Theorem 7.2 (Unconditional MTM over number fields; status **S**, after Levine, Deligne—Goncharov, Bloch–Kriz)**.** *Let k be a number field, or the ring $\mathcal{O}_{k,S}$ of S-integers for a finite set of places S. Then the category* $\mathrm{MTM}(k)$ *of mixed Tate motives unramified outside S is a neutral Tannakian category, unconditionally constructed — independent of the standard (still open) conjectures on general motives — via an explicit differential graded algebra model built from Bloch's higher Chow groups (the Bloch–Kriz construction). For $k = \mathbb{Q}$ (unramified over $\mathbb{Z}$) this category is generated by the motivic fundamental group of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, whose periods are the multiple zeta values (Brown); a general number field k requires the cyclotomic variants of that fundamental group (punctures at roots of unity) or, equivalently, the full higher-Chow DGA. Its Tannaka group is an extension*

$$1 \;\to\; U \;\to\; G_{\mathrm{MTM}(k)} \;\to\; \mathbb{G}_m \;\to\; 1,$$

where $\mathbb{G}_m$ acts by the weight grading and U is pro-unipotent with graded Lie algebra free on generators dual to $\mathrm{Ext}^1_{\mathrm{MTM}(k)}(\mathbb{Q}(0), \mathbb{Q}(n))$*.*

Discussion and reference. The construction is due to Bloch–Kriz [4], with the Tannakian and fundamental-group formulation developed by Deligne–Goncharov [5] and Levine; Brown's work over $\mathbb{Z}$ [6] is the arithmetic capstone. The extension structure of $G_{\mathrm{MTM}(k)}$ is what makes Rung IV's Goncharov Lie coalgebra — built as the graded dual of the Lie algebra of the pro-unipotent radical U — an actually-constructed object rather than a conjectural one. The freeness of the graded Lie algebra of U, which underlies Rung IV's structural theorem, is a consequence of Borel's computation of the ranks of $\mathrm{Ext}^1_{\mathrm{MTM}(k)}(\mathbb{Q}(0), \mathbb{Q}(n)) \cong K_{2n-1}(k) \otimes \mathbb{Q}$ (dimensions governed by Borel's regulator, the subject of Rung V). We take the theorem as a citable input. □

Remark 7.3 (Why unconditionality matters here; status **H**). For a physics-facing reader the significance of Theorem 7.2 is that the pre-numerical object we are invoking is not vaporware. One does not need to assume the standard conjectures to have a well-defined Tannakian category housing the motivic lifts of multiple polylogarithms and multiple zeta values. The "object behind the number" is, in the mixed Tate world relevant to most known amplitude computations, an object one can actually construct and compute with.

7.3 The Kummer motive: $\log x$ as a period

We now work in full the smallest nontrivial example, which is also one of the three load-bearing worked examples threaded through the whole library.

Example 7.4 (Kummer motive; status **S**)**.** Let $X = \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$ over a field $k \subset \mathbb{C}$ and fix $x \in k^\times$. Consider the relative *homology* motive

$$M \;:=\; H_1\bigl(\mathbb{G}_m,\; \{1, x\}\bigr),$$

the motive of $\mathbb{G}_m$ relative to the two points 1 and x. (We take homology rather than cohomology precisely so that the period pairing below evaluates a de Rham *cohomology* class on a Betti *homology* cycle, and so that the weight filtration comes out as $-2, 0$ in the normalization of Theorem 5.3.) We compute its period.

De Rham side. The period pairing evaluates algebraic de Rham cohomology classes on the Betti homology of the pair. $H^1_{\mathrm{dR}}(\mathbb{G}_m)$ is one-dimensional, spanned by the logarithmic form $\frac{dt}{t}$; the relative pair adds the weight-0 unit class e_0 dual to the 0-cycle $\{1, x\}$. A convenient dual pairing basis is thus indexed by $\{\, [\frac{dt}{t}],\ e_0 \,\}$.

Betti side. H_1 of the pair is spanned by the class of a path γ from 1 to x in $\mathbb{C}^\times$, together with the class of the small loop around 0. A basis of $\mathrm{Real}_{\mathrm{B}}(M)$ is $\{\, [\gamma],\ [\mathrm{loop}] \,\}$.

The period. Pairing the form $\frac{dt}{t}$ with the path γ gives

$$\mathrm{per}(M) \ = \ \int_\gamma \frac{dt}{t} \ = \ \log x. \tag{8}$$

The loop around 0 pairs with $\frac{dt}{t}$ to give $2\pi i$, the period of the Tate twist $\mathbb{Q}(1)$ sitting inside M. Thus the period matrix of M is, in the chosen bases,

$$\begin{pmatrix} 1 & 0 \\ \log x & 2\pi i \end{pmatrix},$$

whose upper-left 1 records the motivic-weight-0 part $\mathbb{Q}(0)$, whose lower-right $2\pi i$ records the motivic-weight-(-2) part $\mathbb{Q}(1)$, and whose off-diagonal entry $\log x$ records the nontrivial *extension* gluing them.

Proposition 7.5 (The Kummer motive is a classified extension; status **S**). *Throughout this subsection the base field is $k = F$, i.e. Rung I's kinematic domain regarded as a number field. The motive $M = H_1(\mathbb{G}_m, \{1, x\})$ of Theorem 7.4 is a nonsplit extension*

$$0 \ \to \ \mathbb{Q}(1) \ \to \ M \ \to \ \mathbb{Q}(0) \ \to \ 0,$$

and the assignment $x \mapsto [M]$ induces a canonical isomorphism of abelian groups

$$\mathrm{Ext}^1_{\mathrm{MTM}(F)}\big(\mathbb{Q}(0), \mathbb{Q}(1)\big) \ \cong \ F^\times \otimes \mathbb{Q}, \tag{9}$$

under which the class attached to $x \in F^\times$ has period $\log x$. The extension is split if and only if x is a root of unity (so $\log x \in 2\pi i\,\mathbb{Q}$, i.e. zero in $F^\times \otimes \mathbb{Q}$).

Proof. Weight considerations force any extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ in $\mathrm{MTM}(F)$ to be a Kummer motive of the above shape; the weight filtration has $\mathrm{gr}^W_0 M = \mathbb{Q}(0)$ and $\mathrm{gr}^W_{-2} M = \mathbb{Q}(1)$ (with the normalization of Theorem 5.3). The extension class is precisely the obstruction to a motivic splitting, and Yoneda's description of Ext^1 as isomorphism classes of such extensions gives a homomorphism $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1)) \to$ (Kummer motives). That this is an isomorphism onto $F^\times \otimes \mathbb{Q}$ is the motivic incarnation of Kummer theory: H^1 of $\mathbb{G}_m$ with its Galois/Hodge structure computes $F^\times \otimes \mathbb{Q}$, and multiplicativity $\log(xy) = \log x + \log y$ matches the group law on extensions (Baer sum). Splitting exactly when $\log x \in 2\pi i\,\mathbb{Q}$ is the statement that x is a root of unity. See [5, 2]. $\square$

Remark 7.6 (The ground floor of the hierarchy; status **H**). Equation (9) is the precise sense in which Rung I's raw ratios $F^\times$ are not merely "inputs" to the later machinery but are *literally* the irreducible weight-one building blocks — the primitives — of the entire coalgebraic edifice that Rung IV erects. As pure mathematics (9) is status **S**; the physical reading "the domain of observable ratios is the ground floor of the whole informational hierarchy" is status **H**. In the functional-essence language (Theorem 6.1), $\log x$ is a numerical shadow of the Kummer motive, and the Kummer motive is the pre-numerical source common to every way of presenting $\log x$.

7.4 The motivic fundamental group: multiple polylogarithms as periods

Example 7.7 (Periods of $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$; status **S**)**.** The single-variable multiple polylogarithms

$$\mathrm{Li}_{n_1,\ldots,n_d}(x) \;=\; \sum_{0<k_1<\cdots<k_d} \frac{x^{k_d}}{k_1^{n_1}\cdots k_d^{n_d}}$$

of Rung II are periods of the motivic fundamental group $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ [8]. (More precisely, since the iterated integration runs from 0 to a variable endpoint x, these are periods of the motivic fundamental *groupoid* with a tangential basepoint at 0; the fundamental group is recovered as the automorphisms of a single fibre, and the mixed Hodge structure on this object is due to Hain [9].) Concretely, the pro-unipotent de Rham fundamental group of the thrice-punctured line has coordinate ring the shuffle algebra on two generators $\{\omega_0, \omega_1\}$ with $\omega_0 = \frac{dt}{t}$, $\omega_1 = \frac{dt}{1-t}$; iterated integration of words in ω_0, ω_1 along the straight path from 0 to x produces exactly these single-variable multiple polylogarithms (hyperlogarithms), and the comparison with the Betti fundamental group realizes them as periods. The multiple zeta values arise as their regularized values at $x = 1$. The fully multivariable multiple polylogarithms $\mathrm{Li}_{n_1,\ldots,n_d}(x_1, \ldots, x_d)$ are not periods of the thrice-punctured line alone: they require the motivic fundamental groups of configuration spaces $\mathcal{M}_{0,n}$ (equivalently, of $\mathbb{P}^1$ with additional punctures), a larger but structurally parallel object. In all cases the motivic lifts $\mathrm{Li}^{\mathfrak{m}}_{n_1,\ldots,n_d}$ generate the mixed Tate part of the Hopf algebra $\mathcal{H}$ on which Rung IV's coproduct acts.

Proposition 7.8 (Weight-one periods are logarithms; status **S**)**.** *The weight-one part of the mixed Tate story reduces to Theorem 7.4: the only weight-one framed mixed Tate motives over F are the Kummer motives, their periods are the logarithms $\log x$ for $x \in F^\times$, and $\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q}$ by Theorem 7.5. Consequently the weight-one primitives of the library's coalgebra are canonically $F^\times \otimes \mathbb{Q}$, a fact that Rung IV promotes to the base case of its structural induction.*

Proof. Weight one cannot split as a sum of two positive weights, so a weight-one framed mixed Tate motive is an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$ up to twist; these are exactly the Kummer motives of Theorem 7.5, with periods $\log x$. The identification of primitives with $F^\times \otimes \mathbb{Q}$ is then (9). ∎

Example 7.9 (The dilogarithm motive; status **S**)**.** The classical dilogarithm $\mathrm{Li}_2(x) = \sum_{k\geq 1} x^k/k^2$ [12] is the weight-two period attached to a framed mixed Tate motive $\mathrm{Li}_2^{\mathfrak{m}}(x)$ whose period matrix, in a suitable basis, takes the block form

$$\begin{pmatrix} 1 & 0 & 0 \\ -\log(1-x) & 2\pi i & 0 \\ \mathrm{Li}_2(x) & 2\pi i \log x & (2\pi i)^2 \end{pmatrix},$$

a unipotent lower-triangular matrix with $1, 2\pi i, (2\pi i)^2$ on the diagonal recording the Tate twists $\mathbb{Q}(0), \mathbb{Q}(1), \mathbb{Q}(2)$ (of motivic weights $0, -2, -4$) and the entries $\log(1-x)$, $\log x$, $\mathrm{Li}_2(x)$ recording the successive extensions. The presence of $\log(1-x)$ and $\log x$ in the first column is the motivic origin of the branch loci $x = 1$ and $x = 0$ of the dilogarithm; that these two weight-one logarithms are the "constituents" of Li_2 is exactly what Rung IV's cobracket $\delta\ \mathrm{Li}_2^{\mathfrak{m}}(x) = \log(1-x) \wedge \log(x)$ records. Here we only note that the entries are all periods of one motive: the dilogarithm's transcendental value and the two logarithms are shadows of a single pre-numerical object.

Theorem 7.9 previews the payoff of the whole ladder. In Rung II the five-term relation constrains Li_2 evaluated at cross-ratios; in Rung IV the same data becomes the cobracket into $\bigwedge^2$ of weight-one primitives; in Rung V the Bloch–Wigner single-valued version becomes a regulator producing a real number. The present rung's job is only to certify that all of these live over one motive.

8 The Emergent Property: The Pre-Numerical Object Behind Measurement

8.1 Statement of the emergent property

Each rung of the library names one emergent structural property that the level below cannot express. Rung I disciplines a *domain* of values; it cannot say which *functions* live on that domain. Rung II supplies the functions — polylogarithms and iterated integrals — and their weight grading; but it treats the special constants those functions produce (the values $\zeta(3)$, $\log 2$, $\mathrm{Li}_2(1/2)$, and so on) as brute transcendental facts, with no explanation of why the *same* constants recur across unrelated computations. The emergent property of the present rung is exactly that explanation.

> **Emergent property (Rung III).** The specific transcendental numbers of Rung II are *periods*: numerical shadows of one underlying geometric-arithmetic object, the motive. Physics observes the shadow; the motive is the pre-numerical object behind the measurement.

Formally, this is the passage from the number $\mathcal{A}$ to the object $\mathcal{A}^{\mathfrak{m}}$ in the master formula (2), $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$. Rung II wrote down $\mathcal{A}$ as a value of a special function. Rung III asserts, and Theorems 7.1 and 7.2 and Theorems 7.4 and 7.7 substantiate, that $\mathcal{A}$ is the period of a genuine object $\mathcal{A}^{\mathfrak{m}}$ in a Tannakian category, and that the object — not the number — is the primary carrier of the physical information.

8.2 Why the same numbers recur

The recurrence puzzle of Section 1 now has a structural answer, status **S** as mathematics and **H** in its physical reading.

Proposition 8.1 (Recurrence via shared motives; status **S/H**). *Suppose two period data $\Pi_1 = (X_1, D_1, \omega_1, \Gamma_1)$ and $\Pi_2 = (X_2, D_2, \omega_2, \Gamma_2)$ have motives $M_1 = \mathrm{Mot}(X_1, D_1)$, $M_2 = \mathrm{Mot}(X_2, D_2)$ that share a common subquotient motive N in $\mathrm{MTM}(F)$. Then the periods of N appear as $\mathbb{Q}$-linear combinations among both $\mathrm{per}(\Pi_1)$ and $\mathrm{per}(\Pi_2)$. In particular, two physically unrelated Feynman integrals whose motivic lifts both contain the Tate motive $\mathbb{Q}(n)$ (or a common polylogarithm motive) will exhibit the same constants (π^{2n}, the same ζ-values, and so on) in their evaluations.*

Proof. Periods are functorial: a morphism of motives $N \hookrightarrow M_i$ induces, via the comparison isomorphism (6) and Theorem 7.1, an inclusion of period matrices, so the period matrix of N is a submatrix (up to $\mathbb{Q}$-linear change of basis) of that of M_i. Hence the numbers realizing N occur in both $\mathrm{per}(\Pi_1)$ and $\mathrm{per}(\Pi_2)$. The physical reading — that this is *why* amplitudes share constants — is status **H** because it presumes each amplitude admits a motivic lift, which is established for mixed Tate cases (e.g. ϕ^4 periods) but open in general [7, 6]. □

The mathematical content of Theorem 8.1 is modest — functoriality of periods — but the shift in viewpoint is the whole point of the rung. Before Rung III, "$\zeta(3)$ appears again" is a coincidence to be checked case by case. After Rung III, it is the visible trace of a shared pre-numerical object, and the right question becomes "*which* motive is common, and what is its structure?" — a question Rung IV answers with the coproduct.

8.3 Measurement as realization

Under the functional-essence reading (Theorem 6.1), a measurement is the evaluation of the realization chain (7): an experiment fixes a kinematic configuration (Rung I), the theory attaches an amplitude object $\mathcal{A}^{\mathfrak{m}}$ (Rung III), and the number the experiment reports is $\mathrm{per}(\mathcal{A}^{\mathfrak{m}})$ (Rung V). The motive is prior to the number in the strong sense that the same motive supports *all* of the number's presentations: its de Rham integrand, its Betti integration cycle, its Galois symmetries. The number is the value of one pairing; the object is what makes all the pairings cohere.

Epistemic status of the emergent property. The mathematical statements underpinning this section — Theorems 7.1 and 7.2, Theorems 7.5 and 8.1 — are status **S**. The claim that *physical measurement in general* is the realization of a motivic object is status **H**, and its strongest ontological form ("physical reality is the realization layer of structured mathematical information") is status **P**. By the monotonicity rule (Theorem 3.1) the composite claim inherits the weakest link: we assert the mathematics as **S** and the universal physical reading as **H**, never upgrading the latter.

9 The Non-Tate Boundary

The tame story of this paper — everything expressible through mixed Tate motives, hence through logarithms, polylogarithms, and multiple zeta values — has a precise boundary, and honesty requires marking it here even though its full statement and proof belong to Rung IV.

Proposition 9.1 (Non-Tate obstruction, forward statement; status **S**)**.** *There exist motivic amplitude objects whose motivic lift contains, as a subquotient, the motive $H^1(E)$ of an elliptic curve E/k with* $\mathrm{End}(E) = \mathbb{Z}$ *(or, more generally, a non-Tate simple motive). For such an object:*

1. *its motivic lift is* not *a comodule element over the mixed Tate Hopf algebra of iterated integrals of $d\log$-forms;*

2. *the weight-graded polylogarithmic anatomy of the tame theory is unavailable: the period matrix contains the elliptic periods ω_1, ω_2 and quasi-periods η_1, η_2 of E, not $\mathbb{Q}$-linear combinations of multiple zeta values;*

3. *one must replace the mixed Tate Hopf algebra by an elliptic-polylogarithm coalgebra, over which the decomposition machinery is restored.*

Sketch; full proof is Rung IV's. The simple objects of $\mathrm{MTM}(k)$ are only the Tate twists $\mathbb{Q}(n)$, of Hodge type (p, p); but $H^1(E)$ is simple of rank two with $h^{1,0} = h^{0,1} = 1$, hence not Tate. Its period matrix $\left(\begin{smallmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{smallmatrix}\right)$ satisfies the Legendre relation $\omega_1\eta_2 - \omega_2\eta_1 = 2\pi i$, and for E without complex multiplication ω_1 is not a $\mathbb{Q}$-linear combination of powers of π and multiple zeta values — a structural consequence of the non-mixed-Tate Tannakian/Galois type of $H^1(E)$ (its Hodge and ℓ-adic realizations lie outside the Tate sub-Tannakian category), independently of classical transcendence bounds. Were the object a comodule element over the mixed Tate Hopf algebra, its coaction would express all its "anatomy" using only Tate generators, contradicting the presence of $H^1(E)$ with its genuinely modular ($\mathrm{SL}_2(\mathbb{Z})$) monodromy. The correct home is the Hopf algebra of iterated integrals of modular forms on E [10, 11]. □

Remark 9.2 (Status of the boundary; status **S**). Theorem 9.1 is the precise, citable form of the standard limitation "not every physical observable is known to be a period expressible in the tame (mixed Tate) coalgebra." It is *not* a defect of the realization picture of this paper: elliptic and modular motives are perfectly good motives with perfectly good realizations and periods. What fails at the boundary is only the *tameness* that makes the weight-graded polylogarithmic bookkeeping finite-dimensional in each weight. The realization chain (7) survives the boundary; the mixed Tate simplification does not.

10 Placement in the Modular Ladder

We record how this rung receives from Rung II and hands off to Rungs IV and V. The library is modular: each of these hand-offs is a concrete mathematical fact, not a promissory note.

Received from Rung II. Rung II produces the multiple polylogarithms and their special values as the transcendental functions and constants of perturbative computation, graded by weight. Theorem 7.7 is the hand-off: those functions' values are periods of $\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$. Rung III explains *why* particular numbers recur across unrelated calculations (Theorem 8.1): they are shadows of the same pre-numerical source.

Handed to Rung IV. Theorem 7.2 supplies the actually-constructed Hopf algebra $\mathcal{H}$ of motivic periods on which Rung IV's coproduct Δ and cobracket δ act. Without Rung III, the coproduct of Rung IV would be a purely formal operation on an unconstructed object; with it, Δ acts on a real Tannakian object whose primitives are $F^\times \otimes \mathbb{Q}$ (Theorem 7.8).

Handed to Rung V. The period map per of (2) and (5) is the specific realization $\alpha = \mathrm{per}$ that Rung V evaluates against a concrete integration cycle to produce a measurable real (or complex) number. Theorem 5.7's period matrix is exactly the datum Rung V's regulator refines. The closure of the ladder — that the number Rung V produces is itself a new element of Rung I's domain $F^\times$ — is the synthesis rung's concern; here we only note that Theorem 7.4's $\log x$ is manifestly such a number.

$$\begin{array}{ccccccccccc} \text{Rung I} & \longrightarrow & \text{Rung II} & \longrightarrow & \boxed{\text{Rung III}} & \longrightarrow & \text{Rung IV} & \longrightarrow & \text{Rung V} & \longrightarrow & \text{Synth.} \\ F,\ F^\times & & \mathrm{Li}_n,\ \text{iter.} & & \mathrm{Mot}(X),\ \mathrm{Real}_\alpha & & \Delta,\ \delta,\ \mathcal{L} & & \mathrm{per},\ \mathrm{reg.} & & \mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}}) \end{array}$$

11 Discussion and Limitations

11.1 What is and is not claimed

The mathematics assembled here is standard: Theorem 7.1 (Huber), Theorem 7.2 (Bloch–Kriz, Deligne–Goncharov, Levine, Brown), Theorem 7.4/Theorem 7.5 (Kummer theory, classical), and Theorem 7.7 (Goncharov's multiple polylogarithms as periods) are all established results, cited to primary sources. Our contribution is their assembly around the perspective of the pre-numerical object, and the explicit status labelling of the physical reading.

We reiterate the four standing limitations of the library, as they bear on this rung:

1. *Analogy is not theory.* The S/H/P labels are part of the formalism. The strongest claims here are status **S** as mathematics; their universal physical reading is labelled **H**.

2. *Motives are not "the functions of" an object.* The functional-essence reading (Theorem 6.1) is status **H** throughout and must never be presented as Grothendieck's definition (see the caution box after Theorem 5.1).

3. *No single universal functor* Math $\to$ Phys*.* Each realization is a local, channel-specific construction, not a global claim.

4. *Not every observable is a tame period.* Theorem 9.1 (the Non-Tate obstruction) makes this a precise, citable limitation rather than a vague caveat.

11.2 Status statistics of this paper

Following the discipline of Section 3, we tabulate the status labels attached to the paper's numbered mathematical statements and to the dictionary entries most specific to this rung.

Item	**Kind**	**Status**
Theorem 7.1 (mixed realizations)	theorem	**S**
Theorem 7.2 (unconditional MTM)	theorem	**S**
Theorem 7.5 ($\mathrm{Ext}^1 \cong F^\times \otimes \mathbb{Q}$)	proposition	**S**
Theorem 7.8 (weight-one = logs)	proposition	**S**
Theorem 8.1 (recurrence via shared motives)	proposition	**S/H**
Theorem 9.1 (Non-Tate obstruction)	proposition	**S**
Theorem 6.1 (functional essence)	definition	**H**
Theorem 6.2 (realization chain)	definition	**S/H**
Motive as pre-numerical essence	dictionary	**S/H**
Betti / de Rham / Hodge realization	dictionary	**S**
Étale realization	dictionary	**S/H**
Comparison isomorphism	dictionary	**S**
Physical reality as realization layer	ontology	**P**

The distribution is deliberately weighted toward **S**: the rung's job is to supply a *well-founded object*, and well-foundedness is a mathematical matter. The single **P** entry is the library's standing ontological proposal, retained here only in the caution box of Section 8 and never used as a premise in a proof.

11.3 Open questions for downstream work

Two questions flagged in the library's knowledge base bear directly on this rung. First, the precise conditions under which $\mathrm{MTM}(k)$ is well-behaved for a *general* field k (not a number field) remain subtle; the unconditional construction of Theorem 7.2 is a number-field statement. Second, a fully worked, elementary (non-Tannakian) exposition of Huber's comparison functor, built upward from Theorems 7.4 and 7.9, would make the realization picture accessible to physicists without the machinery of triangulated categories; the present paper takes a step in that direction but does not complete it.

12 Conclusion

We have developed the third rung of a modular library: the recognition that the transcendental constants of perturbative physics are periods, the numerical shadows of motives, and that the motive — a pre-numerical object in a Tannakian category — is the primary carrier of physical information. The realization functors (Betti, de Rham, Hodge, étale) and the comparison isomorphism make "two calculations, one number" a functorial fact; the Kummer motive exhibits $\log x$ as a period of a classified extension, and the motivic fundamental group of the thrice-punctured line exhibits the multiple polylogarithms as periods. The emergent property named by the rung is the pre-numerical object behind measurement: the shift from asking "what number is this?" to asking "what object is this the shadow of?"

The next rung supplies the decomposition law that acts on this object — the Hopf-algebraic coproduct and the Goncharov Lie coalgebra's cobracket — turning "what is this amplitude made of?" into a provable statement about iterated cuts terminating at the weight-one primitives $F^\times \otimes \mathbb{Q}$ this paper has identified. The rung after that evaluates the object against a cycle, producing the number an experiment could compare to data, and closing the ladder into a loop.

A Notation Reference

Symbol	**Meaning**
$F,\ F^\times$	field and its multiplicative group (kinematic domain)
k	base field, $k \subset \mathbb{C}$, characteristic zero
$\mathrm{Mot}(X)$	motive of a variety X
$\mathrm{MM}(k),\ \mathrm{MTM}(k)$	mixed motives; mixed Tate motives over k
$\mathbb{Q}(n)$	n-th Tate twist, period $(2\pi i)^n$
Real_α	realization functor, $\alpha \in \{\mathrm{B}, \mathrm{dR}, \mathrm{Hdg}, \text{ét}\}$
comp	comparison isomorphism, (6)
$\mathrm{per}(\Pi) = \int_\Gamma \omega$	period of a period datum $\Pi = (X, D, \omega, \Gamma)$
$\mathcal{A}^{\mathfrak{m}}$	motivic amplitude object; $\mathcal{A} = \mathrm{per}(\mathcal{A}^{\mathfrak{m}})$
$\mathcal{H}$	Hopf algebra of motivic periods (Rung IV)
$\mathcal{L}$	Goncharov Lie coalgebra (Rung IV)
$\pi_1^{\mathrm{mot}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$	motivic fundamental group of the punctured line
$\mathrm{Ext}^1_{\mathrm{MTM}(F)}(\mathbb{Q}(0), \mathbb{Q}(1))$	$\cong F^\times \otimes \mathbb{Q}$, Kummer classes

B The Comparison Isomorphism as "One Number, Two Routes"

We give the smallest fully explicit instance of Theorem 5.6, complementary to Theorem 7.4. Take $M = \mathbb{Q}(1) = H_1(\mathbb{G}_m)$, the homology motive whose period is $2\pi i$. Its de Rham realization pairs against the one-dimensional space $H^1_{\mathrm{dR}}(\mathbb{G}_m)$ spanned by the algebraic class $[\frac{dt}{t}]$; its Betti realization is the one-dimensional $\mathbb{Q}$-vector space spanned by the class $[\sigma]$ of the unit circle $\sigma : \theta \mapsto e^{2\pi i\theta}$, $\theta \in [0, 1]$. The comparison isomorphism (6) is the pairing of $[\frac{dt}{t}]$ against $[\sigma]$, and

$$\int_\sigma \frac{dt}{t} \;=\; \int_0^1 \frac{2\pi i\, e^{2\pi i\theta}}{e^{2\pi i\theta}}\, d\theta \;=\; 2\pi i.$$

Thus the single period of $\mathbb{Q}(1)$ is $2\pi i$, exactly as asserted in Theorem 5.3, and the "two routes" — the algebraic form $\frac{dt}{t}$ and the topological loop σ — meet in the one number

$2\pi i$. Every higher Tate period $(2\pi i)^n$ is the period of $\mathbb{Q}(n) = \mathbb{Q}(1)^{\otimes n}$ by multiplicativity of periods under tensor product, the multiplicativity that Rung V promotes to a theorem about the period pairing.

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