

The Goncharov Lie Coalgebra of Observables: Coproduct, Cobracket, and the Decomposition Law

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Abstract

A physical observable — a scattering amplitude, a Feynman period, an entropy — is not an atom. It carries a canonical decomposition into irreducible informational constituents, and the algebraic device that governs that decomposition is a *Lie coalgebra*. This paper is the core of a modular library that reads the mathematics of periods and motives as the bookkeeping system for how observables break apart. We construct the Goncharov Lie coalgebra $\mathcal{L}_G(F) = \bigoplus_{n \geq 1} \mathcal{L}_G(F)_n$ of a field F as the space of indecomposables $\overline{\mathcal{H}}/\overline{\mathcal{H}}^2$ of a connected graded Hopf algebra $\mathcal{H}$ of motivic periods, equipped with the reduced coproduct Δ' , the cobracket $\delta : \mathcal{L}_G(F)_n \rightarrow \bigoplus_{p+q=n} \mathcal{L}_G(F)_p \wedge \mathcal{L}_G(F)_q$, and the weight grading wt. Our main structural results are: (i) a well-definedness theorem for the cobracket, proved from first principles — δ descends to indecomposables, lands in Λ^2 , and satisfies the co-Jacobi identity dual to Jacobi; (ii) a *cogeneration theorem* showing that the weight-graded primitives $P_n = \ker(\delta|_{\mathcal{L}_G(F)_n})$ cogenerate the whole coalgebra, with a canonical isomorphism $P_1 \cong F^\times \otimes \mathbb{Q}$ realizing weight-one primitives as logarithms; (iii) a *Conditional Amplitude Decomposition* theorem transporting the coaction through any Tannakian realization functor; (iv) a worked computation identifying the discontinuity of an amplitude with its first coaction slot (the one-loop cut via δ); and (v) a *Non-Tate Obstruction* theorem delimiting precisely where the tame polylogarithmic anatomy fails. Throughout we apply a three-valued epistemic status system — **S** (standard mathematics), **H** (heuristic representation), **P** (speculative ontology) — to separate what is proved from what is proposed. The emergent property named by this rung is the *decomposition law of observables*: all amplitude complexity is generated by iterated cobrackets terminating at irreducible primitives — the weight-one logarithms (single cuts) together with the irreducible higher-weight constants (such as odd zeta values) that admit no further cut.

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1 Introduction

1.1 The governing perspective

The thesis threaded through this library is a single sentence.

A Goncharov-style Lie coalgebra is not merely a formal algebraic object; it is a *bookkeeping system* for how physical quantities break apart into irreducible informational constituents.

A physical observable is treated here as the *numerical realization* of a deeper, pre-numerical, structured object — a *motive* — and the question “what is this observable made of?” is answered not by further computation but by a *coalgebraic decomposition law*: a coproduct and cobracket that expose the observable’s primitive, irreducible pieces (cuts, discontinuities, factorization channels). The present paper is the rung of the ladder at which this thesis is made precise. Where the neighbouring modules supply a domain of values, a family of special functions, and a pre-numerical object, and where the module above evaluates that object down to a measurable real number, this module supplies the *operation*: the Goncharov coproduct Δ , the cobracket δ , the Lie coalgebra $\mathcal{L}_G(F)$, and the primitive elements. Its emergent property is the decomposition law itself.

1.2 The modular ladder

This is a *modular* framework, not a unified one. It is organized as a ladder of five rungs plus a synthesis, each a standalone development, each composing with its neighbour to add exactly one structural property that the level below cannot express.

Rung	New structure	Emergent property
I	Field F , group $F^\times$, cross-ratios, moduli $M_{0,n}$	A disciplined domain of observable values
II	Li_n , iterated integrals, weight/depth, the symbol	Transcendental depth = loop order
III	Motives, realizations, comparison isomorphism, mixed Tate motives	The pre-numerical object behind measurement
IV	Hopf algebra $\mathcal{H}$, coproduct Δ, cobracket δ, $\mathcal{L}_G(F)$, primitives	The decomposition law of observables
V	Regulators, Deligne–Beilinson cohomology, period pairing $\int_\Gamma \omega$	Passage from motive to measurable number
Synth.	Recomposition into observable = $\text{per}(\text{motive})$	Measurement as closure of the ladder

The present paper is Rung IV, printed in bold. It is the *core*: every other rung either builds toward it (Rungs I–III supply the domain, functions, and pre-numerical object that get decomposed) or evaluates it back down to a number (Rung V).

1.3 Contributions and outline

We aim for this to be the most mathematically rigorous module of the set. Concretely:

- Section 2 reproduces, verbatim, the mathematics-to-physics dictionary and the three-valued epistemic status system (**S/H/P**) that discipline every claim in the library.
- Section 3 develops the coalgebra/comodule/Hopf-algebra machinery from first principles, culminating in the construction of the space of indecomposables $\mathcal{L} = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2$.
- Section 4 proves the *cobracket well-definedness theorem* (Theorem 4.2): the reduced coproduct descends to indecomposables, its antisymmetrization lands in $\Lambda^2 \mathcal{L}$, and the resulting cobracket satisfies the co-Jacobi identity. This is the technical heart of the paper.
- Section 5 defines the Goncharov Lie coalgebra $\mathcal{L}_G(F)$, records Goncharov’s explicit coproduct on iterated integrals, and relates the symbol to the maximal iteration of δ .
- Section 6 proves the main structural theorems: Conditional Amplitude Decomposition (Theorem 6.1), termination and primitivity of the anatomy tree (Theorem 6.5), and the *cogeneration theorem* (Theorem 6.6) with $P_1 \cong F^\times \otimes \mathbb{Q}$.

- Section 7 carries out the two load-bearing worked examples: the symbol and cobracket of the dilogarithm (Theorem 7.1) with the five-term relation and Bloch group, and the one-loop cut computed as the first coaction slot (Theorem 7.4).
- Section 8 proves the Non-Tate Obstruction theorem (Theorem 8.1), delimiting the tame regime, and Section 9 states the (speculative) cosmic-Galois compatibility and the Connes–Kreimer graph-side realization.
- Section 10 states the emergent *decomposition law of observables*. Section 11 discusses limitations, tabulates status statistics, and lists open problems; Section 12 concludes.

This is a standalone development. All citations are to the external literature; no sibling project is referenced.

2 The Dictionary and the Epistemic Status System

Every module of this library opens with the same translation table and the same discipline for labelling how reliable each translation is. They are reproduced here verbatim; they are part of the formalism, not decoration.

2.1 The mathematics $\rightarrow$ physics dictionary

Mathematics	Physical representation
Field F	Kinematic/coordinate domain of possible values
Element of $F^\times$	Ratio, cross-ratio, scale, observable parameter
Polylogarithm $\text{Li}_n(x)$	Iterated propagation / layered amplitude contribution
Period	Measurable physical number obtained from geometry
Motive	Pre-numerical informational object behind the measurement
Lie algebra	Infinitesimal symmetry / generator structure
Lie coalgebra	Decomposition law of observables
Cobracket δ	Factorization channel, cut, boundary, or information split
Weight grading	Complexity depth / loop order / transcendental depth
Primitive element	Irreducible informational unit
Coproduct	Full hierarchical decomposition
Regulator map	Passage from motivic object to physical real number
Hodge/de Rham realization	Analytic form seen by calculation
Betti realization	Topological/path/integration-cycle form
Period pairing	Physical observable as pairing of form and cycle

The central slogans that recur across the library are:

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observable = realization(motivic information)
coalgebra = grammar of physical decomposition
Physics observes numerical shadows of deeper motivic structures.
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2.2 The S / H / P status system

Every dictionary entry and every claim is tagged with one of three status labels.

Label	Meaning	Canonical example
[S]	Standard use in mathematics or mathematical physics	The Goncharov coproduct on motivic polylogarithms
[H]	Strong heuristic representation-dictionary entry	Cobracket as “factorization channel”
[P]	Speculative philosophical ontology	A universal cosmic Galois group acting on all QFT amplitudes

Composite labels **[S/H]** and **[H/P]** occur where an entry is standard as pure mathematics but heuristic or speculative in its physical reading. We adopt the following.

Definition 2.1 (Status monotonicity, **[S]**). Order the labels by $\mathbf{S} < \mathbf{H} < \mathbf{P}$. If two representation entries or two claims compose, the status of the composite is the *minimum* of the two under this order: a chain of translations is only as reliable as its weakest link. In particular a **P**-labelled step cannot be upgraded to **S** merely by composing it with otherwise-rigorous results.

We will not silently upgrade a status-**H** or status-**P** claim to status-**S**. Where a theorem is standard mathematics but its universal physical reading is heuristic, we state the theorem as **S** and flag the reading as **H** in a separate “physical reading” paragraph.

3 Mathematical Framework: Coalgebras, Comodules, and Indecomposables

We work over $\mathbb{Q}$ throughout; unadorned $\otimes$ means $\otimes_{\mathbb{Q}}$. All algebras are associative, unital, commutative, and graded; all coalgebras are coassociative and counital.

3.1 Coalgebras and comodules

Definition 3.1 (Coalgebra, **[S]**). A *coalgebra* over $\mathbb{Q}$ is a $\mathbb{Q}$ -vector space C equipped with a coproduct $\Delta: C \rightarrow C \otimes C$ and a counit $\varepsilon: C \rightarrow \mathbb{Q}$ such that *coassociativity* $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ and *counitality* $(\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta$ hold. It is *cocommutative* if $\tau \circ \Delta = \Delta$, where $\tau(a \otimes b) = b \otimes a$.

Definition 3.2 (Comodule, **[S]**). A right *comodule* over a Hopf algebra H is a vector space A with a coaction $\rho: A \rightarrow A \otimes H$ satisfying $(\rho \otimes \text{id}) \circ \rho = (\text{id} \otimes \Delta_H) \circ \rho$ and $(\text{id} \otimes \varepsilon_H) \circ \rho = \text{id}$.

Definition 3.3 (Connected graded Hopf algebra of motivic periods, **[S]**). Let $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ be a graded, connected ($\mathcal{H}_0 = \mathbb{Q} \cdot 1$), commutative Hopf algebra over $\mathbb{Q}$, with product m , unit u , coproduct Δ , counit ε , and antipode S , all graded of degree 0. We call the grading the *weight* and write $\text{wt}(x) = n$ for $x \in \mathcal{H}_n$. The *augmentation ideal* is $\overline{\mathcal{H}} = \ker \varepsilon = \bigoplus_{n \geq 1} \mathcal{H}_n$. An element $\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}$ with a distinguished period map $\text{per}(\mathcal{A}^{\mathfrak{m}}) = \mathcal{A}$ (the physical number) is a *motivic amplitude object*.

The physical picture (status **[S/H]**): $\mathcal{H}$ is the ring of pre-numerical values — motivic periods — and $\text{per}: \mathcal{H} \rightarrow \mathbb{C}$ is the realization that produces the number a calculation returns. Weight is transcendental depth, i.e. loop order / functional complexity.

3.2 The reduced coproduct

Definition 3.4 (Reduced coproduct, **[S]**). The *reduced coproduct* is the map $\Delta': \overline{\mathcal{H}} \rightarrow \mathcal{H} \otimes \mathcal{H}$ defined by

$$\Delta'(x) := \Delta(x) - x \otimes 1 - 1 \otimes x, \quad x \in \overline{\mathcal{H}},$$

that is, the full coproduct with its two “trivial” (tadpole) terms removed.

Lemma 3.5 (No tadpole terms, **[S]**). *For every $x \in \overline{\mathcal{H}}$ the reduced coproduct satisfies $\Delta'(x) \in \overline{\mathcal{H}} \otimes \overline{\mathcal{H}}$; that is, both tensor factors of every term of $\Delta'(x)$ have strictly positive weight.*

Proof. Write $\Delta(x) = \sum_i a_i \otimes b_i$ with a_i, b_i homogeneous. Counitality gives $\sum_i \varepsilon(a_i) b_i = x$ and $\sum_i a_i \varepsilon(b_i) = x$. Because $\mathcal{H}$ is graded and connected, ε is projection onto $\mathcal{H}_0 = \mathbb{Q} \cdot 1$, so the terms with $a_i \in \mathcal{H}_0$ sum to $1 \otimes x$ and the terms with $b_i \in \mathcal{H}_0$ sum to $x \otimes 1$. Subtracting these two groups (they overlap only in the term $1 \otimes \varepsilon(x) \otimes \cdots$, which vanishes since $\varepsilon(x) = 0$ for $x \in \overline{\mathcal{H}}$) leaves precisely the terms with $\text{wt}(a_i) \geq 1$ and $\text{wt}(b_i) \geq 1$. Hence $\Delta'(x) \in \overline{\mathcal{H}} \otimes \overline{\mathcal{H}}$. $\square$

Lemma 3.6 (Coassociativity of the reduced coproduct, **[S]**). *On $\overline{\mathcal{H}}$ the reduced coproduct is coassociative: $(\Delta' \otimes \text{id}) \circ \Delta' = (\text{id} \otimes \Delta') \circ \Delta'$ as maps $\overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}^{\otimes 3}$.*

Proof. This is a direct expansion. Coassociativity of Δ gives $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$. Writing $\Delta = \Delta' + (\cdot \otimes 1) + (1 \otimes \cdot)$ on $\overline{\mathcal{H}}$ and expanding both sides, all terms containing a factor 1 cancel in pairs by counitality, leaving the reduced identity on $\overline{\mathcal{H}}^{\otimes 3}$; the surviving terms lie in $\overline{\mathcal{H}}^{\otimes 3}$ by Theorem 3.5 applied slotwise. $\square$

3.3 Indecomposables

Definition 3.7 (Indecomposables / Lie coalgebra of a Hopf algebra, **[S]**). The *decomposables* are $\overline{\mathcal{H}}^2 = \{\sum_i x_i y_i : x_i, y_i \in \overline{\mathcal{H}}\}$, the image of the restricted product $m: \overline{\mathcal{H}} \otimes \overline{\mathcal{H}} \rightarrow \overline{\mathcal{H}}$. The space of *indecomposables* is the graded quotient

$$\mathcal{L} := Q(\mathcal{H}) := \overline{\mathcal{H}} / \overline{\mathcal{H}}^2 = \bigoplus_{n \geq 1} \mathcal{L}_n, \quad \mathcal{L}_n = \overline{\mathcal{H}}_n / (\overline{\mathcal{H}}^2)_n.$$

Write $q: \overline{\mathcal{H}} \twoheadrightarrow \mathcal{L}$ for the quotient map.

The point of Section 4 is that $\mathcal{L}$ carries a canonical Lie *coalgebra* structure, dual to the way indecomposables of a commutative algebra become a Lie algebra.

4 The Cobracket and its Well-Definedness

This section is the technical heart of the paper. We construct the cobracket δ and prove three facts, each a nontrivial step: it descends to indecomposables, it is antisymmetric (lands in $\Lambda^2 \mathcal{L}$), and it satisfies co-Jacobi.

4.1 Construction

Definition 4.1 (Cobracket, **[S]**). Let $\pi: \mathcal{L} \otimes \mathcal{L} \rightarrow \Lambda^2 \mathcal{L}$ be the canonical quotient projection onto the antisymmetric square, $\pi(a \otimes b) = a \wedge b$ (so π already builds in $b \wedge a = -a \wedge b$; no factor of $\frac{1}{2}$ is inserted). The *cobracket* is the composite

$$\delta: \mathcal{L} \xleftarrow{q} \overline{\mathcal{H}} \xrightarrow{\Delta'} \overline{\mathcal{H}} \otimes \overline{\mathcal{H}} \xrightarrow{q \otimes q} \mathcal{L} \otimes \mathcal{L} \xrightarrow{\pi} \Lambda^2 \mathcal{L},$$

which is well defined by Theorem 4.2 below. Concretely, for $\bar{x} = q(x) \in \mathcal{L}$ with $\Delta'(x) = \sum_i a_i \otimes b_i$,

$$\delta(\bar{x}) = \sum_i q(a_i) \wedge q(b_i) \in \Lambda^2 \mathcal{L}.$$

4.2 The well-definedness theorem

Theorem 4.2 (Cobracket well-definedness, **[S]**). *Let $\mathcal{H}$ be a connected graded commutative Hopf algebra and $\mathcal{L} = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2$. Then:*

- (a) (Descent.) *The composite $\pi \circ (q \otimes q) \circ \Delta'$ vanishes on $\overline{\mathcal{H}}^2$, hence factors through q to give a well-defined graded linear map $\delta: \mathcal{L} \rightarrow \Lambda^2 \mathcal{L}$ of degree 0 (weight-preserving), i.e. $\delta(\mathcal{L}_n) \subseteq \bigoplus_{p+q=n, p,q \geq 1} \mathcal{L}_p \wedge \mathcal{L}_q$.*
- (b) (Co-Jacobi.) *δ satisfies the co-Jacobi identity: writing $\delta^{(2)} = (\delta \wedge \text{id}) \circ \delta: \mathcal{L} \rightarrow \Lambda^3 \mathcal{L}$ for the induced map, one has $\delta^{(2)} = 0$. Equivalently, $(\text{id} + \sigma + \sigma^2) \circ (\delta \otimes \text{id}) \circ \delta = 0$ where σ is the cyclic permutation of the three tensor factors.*

Consequently $(\mathcal{L}, \delta)$ is a graded Lie coalgebra.

Proof. **(a) Descent.** We must show that for $x, y \in \overline{\mathcal{H}}$ the element $\pi((q \otimes q)\Delta'(xy))$ vanishes. Since Δ is an algebra homomorphism,

$$\Delta(xy) = \Delta(x) \Delta(y) = (x \otimes 1 + 1 \otimes x + \Delta'x)(y \otimes 1 + 1 \otimes y + \Delta'y).$$

Expanding and subtracting $xy \otimes 1 + 1 \otimes xy$ to form $\Delta'(xy)$, the surviving terms are

$$\begin{aligned} \Delta'(xy) &= \underbrace{x \otimes y + y \otimes x}_{(*)} + \underbrace{(x \otimes 1)\Delta'y + (1 \otimes x)\Delta'y + (y \otimes 1)\Delta'x + (1 \otimes y)\Delta'x}_{(**)} \\ &\quad + \underbrace{(\Delta'x)(\Delta'y)}_{(***)}. \end{aligned}$$

We treat the three groups.

*Group (**).* Write $\Delta'y = \sum_j c_j \otimes d_j$ with $c_j, d_j \in \overline{\mathcal{H}}$ (Theorem 3.5). Then $(x \otimes 1)\Delta'y = \sum_j xc_j \otimes d_j$; its left factors xc_j lie in $\overline{\mathcal{H}}^2$, so $q(xc_j) = 0$ and the whole term dies under $q \otimes q$. The term $(1 \otimes x)\Delta'y = \sum_j c_j \otimes xd_j$ dies because $xd_j \in \overline{\mathcal{H}}^2$. Symmetrically for the two terms involving $\Delta'x$. Hence $(q \otimes q)$ annihilates all of (**).

*Group (***)*. Write $\Delta'x = \sum_i a_i \otimes b_i$, $\Delta'y = \sum_j c_j \otimes d_j$; then $(\Delta'x)(\Delta'y) = \sum_{i,j} a_i c_j \otimes b_i d_j$ with $a_i c_j, b_i d_j \in \overline{\mathcal{H}}^2$, so again $q \otimes q$ annihilates it.

Group ()*. $(q \otimes q)(x \otimes y + y \otimes x) = q(x) \otimes q(y) + q(y) \otimes q(x)$, which is *symmetric*, hence killed by the antisymmetrizing projection π .

Therefore $\pi \circ (q \otimes q) \circ \Delta'(xy) = 0$, so the composite descends to $\mathcal{L}$. Weight preservation is immediate: Δ , Δ' , q , and π are all graded of degree 0, and by Theorem 3.5 every term of Δ' on $\overline{\mathcal{H}}_n$ has both factors of weight ≥ 1 summing to n .

(b) Co-Jacobi. First note that the *antisymmetrized* map

$$\tilde{\delta} := (\text{id} - \tau) \circ (q \otimes q) \circ \Delta': \overline{\mathcal{H}} \longrightarrow \mathcal{L} \otimes \mathcal{L}$$

does descend to $\mathcal{L}$, even though the un-antisymmetrized $(q \otimes q) \circ \Delta'$ does not. Indeed, by the computation in part (a) the only part of $(q \otimes q)\Delta'$ that survives on $\overline{\mathcal{H}}^2$ is the *symmetric* term $q(x) \otimes q(y) + q(y) \otimes q(x)$ of Group (*), and this is annihilated by $(\text{id} - \tau)$. Hence $\tilde{\delta}(\overline{\mathcal{H}}^2) = 0$, and $\tilde{\delta}$ induces a map $\mathcal{L} \rightarrow \mathcal{L} \otimes \mathcal{L}$ landing in the antisymmetric tensors, which we identify with $\Lambda^2 \mathcal{L}$ via the canonical projection π of Theorem 4.1; the induced map is exactly the cobracket δ . (This is precisely the point the un-antisymmetrized descent would have gotten wrong: the raw $(q \otimes q)\Delta'$ carries the symmetric residue and does *not* factor through q .)

For co-Jacobi we lift back to $\overline{\mathcal{H}}$, where Δ' is coassociative (Theorem 3.6): $(\Delta' \otimes \text{id})\Delta' = (\text{id} \otimes \Delta')\Delta'$. The claim $(\text{id} + \sigma + \sigma^2)(\delta \otimes \text{id})\delta = 0$ (with σ the cyclic permutation of the three factors) is then the classical fact that the antisymmetrization of a coassociative comultiplication is a Lie cobracket — the exact dual, under the pairing with the graded dual, of the statement that the commutator $[a, b] = ab - ba$ of an associative product satisfies the Jacobi identity (Michaelis [13]). Concretely, expanding $(\delta \otimes \text{id})\delta$ through $\delta = (\text{id} - \tau)\Delta'$ and grouping the resulting terms by the cyclic action, coassociativity of Δ' makes the six terms cancel in three antisymmetric pairs; the companion formalization verifies this cancellation term-by-term on the cofree model (see the accompanying code). Hence $(\mathcal{L}, \delta)$ is a graded Lie coalgebra. $\square$

Remark 4.3 (Duality with the Lie algebra of primitives, **[S]**). Theorem 4.2 is the exact dual of a familiar statement: for a graded connected commutative Hopf algebra $\mathcal{H}$, the graded dual $\mathcal{H}^\vee$ (with respect to the finite-dimensional weight pieces) is a cocommutative Hopf algebra, and by Milnor–Moore its primitives $\text{Prim}(\mathcal{H}^\vee)$ form a graded Lie *algebra* whose universal enveloping algebra is $\mathcal{H}^\vee$. Dualizing, $\mathcal{L} = Q(\mathcal{H}) = \text{Prim}(\mathcal{H}^\vee)^\vee$ is the graded Lie *coalgebra* obtained by transposing the Lie bracket. Our first-principles proof avoids invoking Milnor–Moore, so it applies verbatim in the pro-object / ind-finite setting relevant to motivic periods, where the naive dual is subtle.

Remark 4.4 (Why Λ^2 and not $\otimes$, **[S]**). The cobracket lands in $\Lambda^2 \mathcal{L}$, not $\mathcal{L} \otimes \mathcal{L}$: the symmetric part of the reduced coproduct comes entirely from products (Group (*)) of the proof), which vanish on indecomposables. This is the coalgebraic shadow of the physical statement that the *ordered* pair of pieces in a single factorization is only defined up to a sign — a cut has two sides, and swapping them costs a sign, not new information.

5 The Goncharov Lie Coalgebra

5.1 Definition

Definition 5.1 (Goncharov Lie coalgebra of a field, **[S]**). Let F be a field of characteristic 0 (typically a number field, or a function field of kinematic invariants). Let $\mathcal{H} = \mathcal{H}(F)$ be the connected graded commutative Hopf algebra of *framed mixed Tate motives over F* — equivalently, the Hopf algebra of motivic iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and its cyclotomic variants, generated by the motivic polylogarithm classes $\mathrm{Li}_n^{\mathrm{m}}(x)$ and their multiple analogues, graded by weight. The *Goncharov Lie coalgebra* is

$$\mathcal{L}_G(F) = \bigoplus_{n \geq 1} \mathcal{L}_G(F)_n = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2,$$

the space of indecomposables of Theorem 3.7, equipped with the weight grading and the cobracket δ of Theorem 4.1, whose well-definedness is Theorem 4.2.

The existence of $\mathcal{H}(F)$ as an *actually constructed* (not conjectural) object over a number field is the theorem of Deligne–Goncharov [11], Bloch–Kriz [19], and Levine, building on the mixed Tate motive formalism; Brown’s work [4] completes the picture over $\mathbb{Z}$. This is precisely the payoff of Rung III of the ladder: without it, the coproduct of the present paper would act on an unconstructed object.

5.2 Goncharov’s explicit coproduct

The Hopf algebra $\mathcal{H}(F)$ carries Goncharov’s combinatorial coproduct on iterated integrals. For $I(a_0; a_1, \dots, a_N; a_{N+1})$ the motivic iterated integral with endpoints a_0, a_{N+1} and interpolation points $a_1, \dots, a_N \in F$,

$$\Delta I(a_0; a_1, \dots, a_N; a_{N+1}) = \sum_{0=i_0 < i_1 < \dots < i_k < i_{k+1}=N+1} \left(\prod_{p=0}^k I(a_{i_p}; a_{i_p+1}, \dots, a_{i_{p+1}-1}; a_{i_{p+1}}) \right) \otimes I(a_0; a_{i_1}, \dots, a_{i_k}; a_{N+1}). \quad (1)$$

The right-hand tensor factor runs over subsequences $\{a_{i_1}, \dots, a_{i_k}\}$ of the interpolation points; the left factor is the product of the “quotient” integrals living on the complementary segments. This combinatorics — choosing which interpolation points survive — is the arithmetic mechanism behind the “cuts” of the physics dictionary (Section 7).

Proposition 5.2 (Coassociativity of (1), **[S]**). *The coproduct (1) is coassociative and, together with the shuffle product and the weight grading, makes $\mathcal{H}(F)$ a connected graded commutative Hopf algebra.*

Proof (Goncharov [2]). Coassociativity is a combinatorial identity: both $(\Delta \otimes \mathrm{id})\Delta$ and $(\mathrm{id} \otimes \Delta)\Delta$ enumerate pairs of nested subsequences $\{a_i\} \subseteq \{a_j\} \subseteq \{a_1, \dots, a_N\}$, with the three tensor factors given by the three segments determined by the flag; the two enumerations agree. Compatibility with the shuffle product and gradedness are Goncharov’s Theorem in [2]; connectedness is $\mathcal{H}_0 = \mathbb{Q}$. $\square$

5.3 The symbol as maximal iteration

Definition 5.3 (Symbol, **[S]**). The *symbol* of a weight- n element $x \in \mathcal{H}_n$ is the image of x under the maximal iteration of the reduced coproduct into weight-one pieces:

$$\mathcal{S}(x) := \underbrace{(\Delta'_{1,1,\dots,1})}_{n \text{ factors}}(x) \in \mathcal{H}_1^{\otimes n} \cong (F^\times \otimes \mathbb{Q})^{\otimes n},$$

where $\Delta'_{1,\dots,1}$ is the component of the $(n-1)$ -fold iterated reduced coproduct landing in the all-weight-one graded piece $\mathcal{H}_1^{\otimes n}$.

The symbol is the sharpest available invariant for detecting functional identities among polylogarithms; Theorem 7.1 computes it for the dilogarithm. The relation to the cobracket is that $\mathcal{S}$ is the total (all-slot) iteration of Δ' , whereas δ retains only the top graded anti-symmetrized slot at each step; the symbol thus refines the cobracket.

6 Main Structural Theorems

6.1 Conditional amplitude decomposition

Theorem 6.1 (Conditional Amplitude Decomposition, **[S]** on a conditional hypothesis). *Let $r_\alpha: \mathcal{H} \rightarrow \mathcal{H}_\alpha$ be a homomorphism of connected graded Hopf algebras induced on motivic-Galois Hopf algebras by a geometric Tannakian realization functor Real_α (Betti, de Rham, Hodge, or étale — each with $\mathcal{H}_\alpha$ itself a connected graded Hopf algebra); i.e. r_α intertwines the coproducts,*

$$(r_\alpha \otimes r_\alpha) \circ \Delta = \Delta_\alpha \circ r_\alpha.$$

Then for every motivic amplitude object $\mathcal{A}^m \in \mathcal{H}$ with coaction $\Delta(\mathcal{A}^m) = \sum_i \mathcal{A}_{i,1}^m \otimes \mathcal{A}_{i,2}^m$, the realized amplitude $r_\alpha(\mathcal{A}^m)$ carries the decomposition

$$\Delta_\alpha(r_\alpha(\mathcal{A}^m)) = \sum_i r_\alpha(\mathcal{A}_{i,1}^m) \otimes r_\alpha(\mathcal{A}_{i,2}^m), \quad (2)$$

natural in $\mathcal{A}^m$ and compatible with the weight and depth gradings.

Remark 6.2 (The period realization is a comodule coaction, not a Hopf map, **[S]**). The physically relevant “period realization” is *not* an instance of Theorem 6.1: the numerical period map $\text{per}: \mathcal{H} \rightarrow \mathbb{C}$ lands in the ring of complex numbers, which carries no weight grading and no nontrivial coproduct, so per is not a homomorphism of graded Hopf algebras. The correct structure is a *comodule*. The ring of (motivic) periods $\mathcal{P}$ is a right $\mathcal{H}$ -comodule with coaction $\rho: \mathcal{P} \rightarrow \mathcal{H} \otimes \mathcal{P}$ (Brown’s coaction [6, 7]), and ρ is a comodule morphism intertwining the coproduct of $\mathcal{H}$ with the coaction of $\mathcal{P}$:

$$(\Delta \otimes \text{id}_{\mathcal{P}}) \circ \rho = (\text{id}_{\mathcal{H}} \otimes \rho) \circ \rho.$$

The decomposition (2) then holds in the comodule form $\rho(\text{per}(\mathcal{A}^m)) = \sum_i \mathcal{A}_{i,1}^m \otimes \text{per}(\mathcal{A}_{i,2}^m)$ (or its de Rham/Betti bigraded refinement), which is exactly Brown’s coaction principle on Feynman periods and the Goncharov coproduct on polylogarithms. Theorem 7.4 below is the concrete numerical shadow of this comodule coaction.

Proof. This is a diagram chase. Along the top-right path, bilinearity gives

$$(r_\alpha \otimes r_\alpha)(\Delta \mathcal{A}^{\mathfrak{m}}) = (r_\alpha \otimes r_\alpha) \left(\sum_i \mathcal{A}_{i,1}^{\mathfrak{m}} \otimes \mathcal{A}_{i,2}^{\mathfrak{m}} \right) = \sum_i r_\alpha(\mathcal{A}_{i,1}^{\mathfrak{m}}) \otimes r_\alpha(\mathcal{A}_{i,2}^{\mathfrak{m}}),$$

while along the left-bottom path it equals $\Delta_\alpha(r_\alpha(\mathcal{A}^{\mathfrak{m}}))$. Equating the two paths using the intertwining hypothesis gives (2). Existence of r_α as a graded Hopf-algebra homomorphism is the statement that a Tannakian realization functor induces a homomorphism of the associated motivic-Galois Hopf algebras (functoriality of the Tannakian dual, Rung III). $\square$

Remark 6.3 (Where the S/H/P labels bite, **[H]**). As pure algebra, Theorem 6.1 is a formal property of coproduct-compatible Hopf homomorphisms — always true once the hypotheses hold (**[S]**). Its physical content lies entirely in the antecedent $\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}$: whether a given physical amplitude actually admits a motivic lift inside the Hopf algebra of motivic periods. For Feynman integrals evaluating to multiple zeta values or mixed Tate polylogarithms this is established (**[S]**; e.g. Panzer–Schnetz on ϕ^4 periods [9]). For amplitudes requiring elliptic or higher motivic structures the comodule hypothesis fails over the Tate Hopf algebra (Theorem 8.1); the decomposition is then only heuristic (**[H]**) until one passes to the appropriate larger coalgebra.

6.2 Termination and primitivity

Definition 6.4 (Coalgebraic anatomy, primitives, **[S]**). Given $x \in \mathcal{L}$, its *coalgebraic anatomy* is the rooted tree obtained by iterating the reduced coaction: the root is x ; the children are the tensor factors of $\tilde{\delta}(x)$; and one recurses on each factor. A *primitive* is an x with $\delta(x) = 0$, equivalently $\tilde{\delta}(x) = 0$; the primitives are the leaves of every anatomy tree. Write $P_n = \ker(\delta|_{\mathcal{L}_n})$ for the weight- n primitives and $P = \bigoplus_n P_n$.

Proposition 6.5 (Termination and primitivity of the anatomy tree, **[S]**). *For a motivic amplitude object $\mathcal{A}^{\mathfrak{m}} \in \mathcal{H}_n$ of weight n , the iterated reduced coaction terminates after at most n steps, and every terminal piece is a primitive. Every primitive has weight between 1 and n ; the weight-one primitives are logarithms (Theorem 6.6(b)), but higher-weight primitives also occur — for instance $\zeta^{\mathfrak{m}}(3) \in \mathcal{H}_3$ satisfies $\Delta' = 0$ and is a weight-three primitive over $\mathbb{Q}$. A branch of the anatomy tree stops as soon as it reaches any primitive, not only when it reaches weight one.*

Proof. By Theorem 3.5 every term of $\Delta'(\mathcal{A}^{\mathfrak{m}})$ has both factors of weight ≥ 1 summing to n ; hence each factor has weight $\leq n - 1$, and each iteration of Δ' strictly lowers the weight of every factor. Since weights are positive integers bounded by n , after at most $n - 1$ further steps each branch reaches a factor on which Δ' vanishes — by definition a primitive. At weight 1 this is automatic (Δ' cannot split $1 = p + q$ with $p, q \geq 1$, so $\delta|_{\mathcal{L}_1} = 0$), but Δ' may also vanish at higher weight (e.g. $\zeta^{\mathfrak{m}}(3)$), in which case the branch terminates there. Finiteness of the tree follows since each node has finitely many children (the weight pieces are finite-dimensional over F -arithmetic data) and depth is bounded by n . $\square$

6.3 The cogeneration theorem

This is the structural anchor of the paper: the primitives are the irreducible informational units, and they cogenerate everything.

Theorem 6.6 (Weight-graded primitives cogenerate $\mathcal{L}_G(F)$, **[S]**). *Let F be a number field and $\mathcal{L}_G(F) = \bigoplus_n \mathcal{L}_G(F)_n$ with cobracket δ and primitives $P_n = \ker(\delta|_{\mathcal{L}_G(F)_n})$. Then:*

- (a) (Cogeneration.) $P = \bigoplus_{n \geq 1} P_n$ cogenerates $\mathcal{L}_G(F)$: the intersection of the kernels of all iterated reduced coactions is 0, so every element is recovered from its iterated cobrackets, whose leaves lie in P (the primitives of all weights).
- (b) (Weight-one primitives are logarithms.) There is a canonical weight-one isomorphism $P_1 = \mathcal{L}_G(F)_1 \cong F^\times \otimes \mathbb{Q}$ that sends $x \in F^\times$ to the Kummer class whose period is $\log x$.
- (c) (Free cogeneration.) The graded dual of $(\mathcal{L}_G(F), \delta)$ is a graded Lie algebra which, for a number field F , is free on generators dual to $\text{Ext}_{\text{MTM}(F)}^1(\mathbb{Q}(0), \mathbb{Q}(n))$ in each weight n . Equivalently the primitives $P_n \cong \text{Ext}_{\text{MTM}(F)}^1(\mathbb{Q}(0), \mathbb{Q}(n))^\vee$ are the weight- n cogenerators; they are nonzero in infinitely many weights (over $\mathbb{Q}$, $\dim_{\mathbb{Q}} P_n = 1$ for n odd ≥ 3 , generated by $\zeta^{\mathfrak{m}}(n)$, and 0 for n even ≥ 2). The associated graded of $\mathcal{L}_G(F)$ under the coradical filtration is spanned by iterated cobrackets of these primitives.

Remark 6.7 (Cogeneration is by all primitives, not by P_1 alone, **[S]**). It is essential that the cogenerators are $P = \bigoplus_n P_n$, not merely the weight-one part P_1 . The dual Lie algebra of Theorem 6.6(c) is free on generators in each odd weight ≥ 3 , so it is *not* generated by its weight-one elements alone; dually, higher-weight primitives (such as $\zeta^{\mathfrak{m}}(3)$) are genuine irreducible informational units that no iterated cobracket of logarithms can reach. The physical reading (Theorem 6.8) must therefore include these irreducible constants alongside the weight-one cuts.

Proof. (b) Weight-one framed mixed Tate motives over F are extensions of $\mathbb{Q}(0)$ by $\mathbb{Q}(1)$, classified by

$$\mathcal{L}_G(F)_1 \cong \text{Ext}_{\text{MTM}(F)}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \cong F^\times \otimes \mathbb{Q},$$

the Kummer motives (Rung III): the extension attached to $x \in F^\times$ has de Rham class dt/t , Betti class a path from 1 to x , and period $\int_1^x dt/t = \log x$. Since weight one cannot split as $p + q$ with $p, q \geq 1$, Theorem 6.5 gives $\delta|_{\mathcal{L}_G(F)_1} = 0$, so $P_1 = \mathcal{L}_G(F)_1$.

(c) The graded dual $\mathcal{L}_G(F)^\vee = \bigoplus_n (\mathcal{L}_G(F)_n)^\vee$ is the motivic Galois Lie algebra $\mathfrak{g} = \text{Lie}(\pi_1(\text{MTM}(F)))$ of the pro-unipotent radical of the Tannakian fundamental group of $\text{MTM}(F)$. By Deligne–Goncharov [11] and Borel’s computation of the ranks of K -groups of number fields [20], for a number field F the graded pieces satisfy

$$\dim_{\mathbb{Q}} (\mathfrak{g}^{\text{ab}})_n = \dim_{\mathbb{Q}} \text{Ext}_{\text{MTM}(F)}^1(\mathbb{Q}(0), \mathbb{Q}(n)) = \begin{cases} r_1 + r_2 & n \text{ odd} > 1, \\ r_2 & n \text{ even}, \\ r_1 + r_2 & n = 1 \text{ (i.e. rank } F^\times \otimes \mathbb{Q}), \end{cases}$$

and there are no relations forced beyond those visible on generators; equivalently $\mathfrak{g}$ is a *free* graded Lie algebra on a set of generators indexed weightwise by these Ext groups. A free

graded Lie algebra is generated by its weightwise generators, so its graded dual coalgebra $\mathcal{L}_G(F)$ is *cogenerated* by the corresponding primitives.

(a) A graded connected coalgebra carries the coradical (primitive) filtration $F_0 \subseteq F_1 \subseteq \dots$, with $F_0 = P$ the primitives and $F_k = \{x : \tilde{\delta}(x) \in \sum_{i+j=k-1} F_i \otimes F_j\}$. The intersection $\bigcap_k \ker(\tilde{\delta}^{[k]})$ of kernels of all iterated coactions is 0 in a connected graded coalgebra: an element in the intersection has $\tilde{\delta}$ -image vanishing at every depth, forcing it into $F_0 = P$. Hence every element is reconstructed from its iterated cobrackets; by Theorem 6.5 the iteration terminates after at most n steps, and its leaves are primitives — weight-one logarithms $F^\times \otimes \mathbb{Q}$ by (b), together with the higher-weight primitives of (c). The union $\bigcup_n \bigoplus_{k \leq n} P_k$ of all these primitives is the cogenerating set. $\square$

Remark 6.8 (Physical reading, **[H]**). Under the dictionary, weight = transcendentality = loop/functional complexity, and cobracket = single discontinuity/cut. Theorem 6.6 then reads: *all amplitude complexity is generated by iterated cobrackets of two kinds of irreducible piece* — the weight-one logarithmic building blocks (each a single cut) *and* the irreducible higher-weight constants (such as $\zeta(3)$) that have vanishing cobracket and therefore admit no further cut. This is a candidate rigorous counterpart of the physics folklore that multi-loop symbols are built from iterated discontinuities down to logarithms and transcendental constants. The status is **H** because the identification “cobracket = physical cut” is a representation-dictionary entry, not a theorem of physics; the mathematics of cogeneration is **S**.

7 Two Load-Bearing Worked Examples

We now compute the two examples that connect this rung to its neighbours: the dilogarithm (which threads Rungs II, IV, V through the five-term relation and Bloch group) and the one-loop cut (the paradigm instance of “cobracket = cut”).

7.1 The symbol and cobracket of the dilogarithm

Example 7.1 (Dilogarithm, five-term relation, Bloch group, **[S]**). Let $\text{Li}_2^{\text{m}}(x)$ be the motivic dilogarithm and $\log^{\text{m}}(x)$ the motivic logarithm (weight one, $\log^{\text{m}}(x) \leftrightarrow x \in F^\times \otimes \mathbb{Q}$). Its reduced coproduct is

$$\Delta' \text{Li}_2^{\text{m}}(x) = -\text{Li}_1^{\text{m}}(x) \otimes \log^{\text{m}}(x) = \log^{\text{m}}(1-x) \otimes \log^{\text{m}}(x),$$

using $\text{Li}_1(x) = -\log(1-x)$. Hence the symbol is

$$\mathcal{S}(\text{Li}_2(x)) = -(1-x) \otimes x \in (F^\times \otimes \mathbb{Q})^{\otimes 2},$$

and the cobracket is

$$\delta \text{Li}_2^{\text{m}}(x) = \log^{\text{m}}(1-x) \wedge \log^{\text{m}}(x) \in \Lambda^2 \mathcal{L}_G(F)_1.$$

The two tensor slots are the two weight-one primitives — physically the two branch loci $x = 1$ and $x = 0$ of the dilogarithm, i.e. its cut structure.

The five-term relation as a cobracket kernel. The Rogers dilogarithm $L(x) = \text{Li}_2(x) + \frac{1}{2} \log(x) \log(1-x)$ satisfies the five-term relation

$$L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right),$$

and its single-valued avatar, the Bloch–Wigner function $D(z) = \text{im}(\text{Li}_2(z)) + \arg(1-z) \log|z|$, satisfies the clean form

$$D(u) + D(v) + D\left(\frac{1-u}{1-uv}\right) + D(1-uv) + D\left(\frac{1-v}{1-uv}\right) = 0.$$

Proposition 7.2 (Weight-two primitives and the Bloch group, **[S]**). *Let $B(F) = \mathbb{Z}[F \setminus \{0, 1\}]/(\text{five-term relations})$ be the Bloch group. The weight-two part of $\mathcal{L}_G(F)$ is, up to isogeny, identified with $B(F) \otimes \mathbb{Q}$; under this identification the cobracket is the map*

$$\delta: \mathcal{L}_G(F)_2 \longrightarrow \Lambda^2(F^\times \otimes \mathbb{Q}), \quad \{x\} \longmapsto (1-x) \wedge x,$$

and its kernel P_2 consists exactly of the $\mathbb{Q}$ -combinations of Bloch–Wigner symbols satisfying the five-term relation. Consequently the five-term relation is precisely the statement that a distinguished alternating sum of weight-two elements is a primitive (has vanishing cobracket).

Proof. Define $\{x\} \in \mathcal{L}_G(F)_2$ as the class of $\text{Li}_2^\mathfrak{m}(x)$. By Theorem 7.1, $\delta\{x\} = (1-x) \wedge x$ in $\Lambda^2(F^\times \otimes \mathbb{Q}) = \Lambda^2 \mathcal{L}_G(F)_1$ (Theorem 6.6(b)). A direct computation — expand $(1-\xi) \wedge \xi$ for each of the five cross-ratio arguments and use bilinearity and antisymmetry of $\wedge$ — shows the five-term combination $\sum (-1)^i \{[\cdots]\}$ maps to 0 under δ ; this is the well-known verification that the Bloch–Wigner symbol respects the five-term relation at the level of Λ^2 . Hence the five-term combination lies in $\ker \delta = P_2$. Zagier–Bloch–Suslin [21, 23] identify $\ker(\delta|_{\mathcal{L}_G(F)_2})$ with the relations defining $B(F) \otimes \mathbb{Q}$, giving the stated identification. $\square$

Remark 7.3 (Forward pointer to the regulator, **[S]**). Theorem 7.2 is exactly what licenses Rung V’s Bloch–Wigner regulator: the well-definedness of $D: B(F) \rightarrow \mathbb{R}$ rests on D respecting the five-term relation, i.e. on the vanishing, inside $\mathcal{L}_G(F)$, of the cobracket combination $\sum (-1)^i D([\cdots])$. The regulator’s very existence is a shadow of a cobracket kernel computed here.

7.2 The one-loop cut as the first coaction slot

Here $\text{Disc}_s \mathcal{A} := \mathcal{A}(se^{2\pi i}) - \mathcal{A}(s)$ denotes the discontinuity of the multivalued function $\mathcal{A}$ across the branch cut emanating from $s = 0$, i.e. the jump induced by the Picard–Lefschetz monodromy operator that encircles the vanishing locus of the kinematic invariant s ; it is the numerical incarnation of the comodule coaction of Theorem 6.2.

Proposition 7.4 (Discontinuity computes the first coaction slot, **[S]**). *Let $\mathcal{A}^\mathfrak{m} \in \mathcal{H}$ be a motivic amplitude object with $\Delta'(\mathcal{A}^\mathfrak{m}) = \sum_i \mathcal{A}_{i,1}^\mathfrak{m} \otimes \mathcal{A}_{i,2}^\mathfrak{m}$. Under the period realization, the discontinuity of $\mathcal{A} = \text{per}(\mathcal{A}^\mathfrak{m})$ across the branch point associated to a weight-one primitive $\log^\mathfrak{m} s$ (s a kinematic invariant vanishing on the cut) is*

$$\text{Disc}_s \mathcal{A} = 2\pi i \sum_{i: \mathcal{A}_{i,2}^\mathfrak{m} = \log^\mathfrak{m} s} \text{per}(\mathcal{A}_{i,1}^\mathfrak{m}), \quad (3)$$

i.e. the first tensor factors paired with the second factor $\log^m s$. Iterating on $\mathcal{A}_{i,1}^m$ recovers the full sequence of multiple discontinuities as deeper coaction slots.

Proof. The monodromy of $\text{per}(\mathcal{A}^m)$ around $s = 0$ acts on the Betti realization by the Picard–Lefschetz transformation, whose logarithm is the weight-one nilpotent attached to $\log^m s$. Under the motivic coaction the monodromy operator is dual to the second tensor slot: Δ' intertwines analytic continuation with the H -comodule structure (Rung III), so the variation $\mathcal{A}^m \mapsto \mathcal{A}^m(se^{2\pi i}) - \mathcal{A}^m(s)$ picks out exactly the terms whose second factor is $\log^m s$, with coefficient $2\pi i = \text{per}(\mathbb{Q}(1))$. Realizing gives (3); iterating on the first factor $\mathcal{A}_{i,1}^m$ (itself a lower-weight amplitude object) yields the multiple discontinuities as iterated first slots. $\square$

Example 7.5 (One-loop cut, **[S/H]**). For the one-loop bubble/triangle contributing a dilogarithm $\text{Li}_2(1 - s/m^2)$, Theorem 7.1 gives $\Delta' = \log^m(s/m^2) \otimes \log^m(1 - s/m^2)$ (up to the arguments' normalization), so Theorem 7.4 returns $\text{Disc}_s \mathcal{A} = 2\pi i \log(s/m^2)$ across the physical threshold. The single cut δ has computed the imaginary part / unitarity discontinuity of the loop integral directly from the coaction, without evaluating the integral. This is the paradigm instance of the dictionary entry “cobracket $\delta \leftrightarrow$ factorization channel, cut, boundary, or information split” made into an actual computation. The mathematics is **S**; the reading of a Feynman cut as “the” physical meaning of δ is **H**.

8 The Non-Tate Obstruction

The tame, weight-graded polylogarithmic anatomy of Theorem 6.6 is not universal. The following limitation theorem makes precise exactly where it fails.

Theorem 8.1 (Non-Tate Obstruction, **[S]**as mathematics). *Let $\mathcal{A}^m$ be a motivic amplitude object whose motivic lift has, as a subquotient, the motive $H^1(E)$ of an elliptic curve E/k with $\text{End}(E) = \mathbb{Z}$ (or, more generally, a non-Tate simple motive). Then:*

- (i) $\mathcal{A}^m$ is not a comodule element over the mixed Tate Hopf algebra H_{MT} of iterated integrals of $d \log$ -forms;
- (ii) the weight-graded polylogarithmic anatomy of Theorem 6.6 is unavailable: the primitives are no longer exhausted by $F^\times \otimes \mathbb{Q}$, and the period matrix contains the elliptic periods ω_1, ω_2 (and quasi-periods η_1, η_2) of E , not $\mathbb{Q}$ -linear combinations of multiple zeta values;
- (iii) one must replace H_{MT} by an elliptic-polylogarithm coalgebra, and Theorem 6.1 holds only relative to that larger Hopf algebra.

Proof. (ii) The simple objects of $\text{MTM}(k)$ are only the Tate twists $\mathbb{Q}(n)$, of Hodge type $(-n, -n)$; $H^1(E)$ is simple of rank two with $h^{1,0} = h^{0,1} = 1$, hence not Tate. Its period matrix $\begin{pmatrix} \omega_1 & \omega_2 \\ \eta_1 & \eta_2 \end{pmatrix}$ has determinant $2\pi i$ (Legendre relation), and by transcendence results ω_1 is not a $\mathbb{Q}$ -linear combination of powers of π and multiple zeta values for E without complex multiplication. (i) If $\mathcal{A}^m$ were a comodule element over H_{MT} , its coaction would express all its anatomy using only Tate generators, contradicting the presence of $H^1(E)$ as a subquotient with genuinely modular ($\text{SL}_2(\mathbb{Z})$) monodromy. (iii) The correct home is the Hopf algebra

of iterated integrals of modular forms/functions on E (Levin–Racinet [17]; Broedel–Duhr–Dulat–Tancredi [16]); over this larger algebra the comodule hypothesis of Theorem 6.1 is restored. $\square$

Remark 8.2 (The precise, citable limitation, **[S]**). Theorem 8.1 is the exact form of the standard caveat “not every physical observable is known to be a period expressible in the tame (mixed Tate) coalgebra.” It converts a vague warning into a theorem with a named obstruction ($H^1(E)$ as subquotient) and a named remedy (elliptic polylogarithms). Status-monotonicity (Theorem 2.1) then forbids advertising the tame decomposition law as universal.

9 Cosmic Galois Compatibility and the Graph Side

9.1 The cosmic Galois group

Proposition 9.1 (Cosmic Galois compatibility, **[H/P]**). *Suppose a cosmic Galois group G_c acts on the Hopf algebra $\mathcal{H}$ of motivic periods by Hopf automorphisms compatible with Δ . Then the action descends through the period realization to an action on the numerical amplitude ring commuting with the coaction-transported decomposition of Theorem 6.1.*

Proof (conditional/heuristic). Formally immediate if $G_c \subseteq \text{Aut}^\otimes(\omega)$, the Tannakian automorphism group of the fibre functor $\omega = \text{per}$, acts by Hopf automorphisms: it then commutes with the comodule structure by definition, and Theorem 6.1 transports this equivariance through $\text{Real}_\alpha = \text{per}$. The entire content is in the antecedent: a precise definition of G_c for a general QFT is open (Brown’s conjecture [6]); verified instances are the coaction on ϕ^4 periods up to high loop order (Panzer–Schnetz [9]), which give status-**S** evidence for special cases without establishing the general statement. $\square$

This is the most speculative claim of the rung; by Theorem 2.1 it is labelled **[H/P]** and may not be upgraded.

9.2 Connes–Kreimer: the decomposition principle on the graph side

The same formal Hopf-algebraic grammar governs the combinatorial side of renormalization.

Theorem 9.2 (Connes–Kreimer realizes the decomposition principle, **[S]**). *For the Connes–Kreimer Hopf algebra H_{CK} of Feynman graphs, graded by loop number, with coproduct*

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma,$$

the antipode $S(\Gamma) = -\Gamma - \sum_{\gamma \subsetneq \Gamma} S(\gamma) \cdot (\Gamma/\gamma)$ computes the BPHZ renormalization counterterm; the renormalized Feynman-rules character is the Birkhoff decomposition $\varphi_R = \varphi_+$ of the loop $\varphi: H_{\text{CK}} \rightarrow \mathbb{C}$ in the group of characters, relative to a Rota–Baxter splitting $\mathbb{C} = \mathbb{C}_+ \oplus \mathbb{C}_-$.

Proof (Connes–Kreimer [10]). (H_{CK}, Δ) , graded by loop number, is connected, hence has a unique antipode making it Hopf. Multiplicativity of φ under the Rota–Baxter splitting yields the BPHZ formula as the Birkhoff decomposition of φ . This is the *same* formal grammar as the Goncharov coproduct (1): where Goncharov’s Δ decomposes the *value* of an integral (period side), Connes–Kreimer’s Δ decomposes the *graph* (combinatorial side); the two are compatible via the period map (Feynman rules = $\text{per} \circ \text{Mot}$), and their compatibility square commutes exactly when the amplitude admits a motivic lift ([S] for mixed Tate cases, e.g. Panzer–Schnetz [9]). $\square$

10 The Decomposition Law of Observables

We now state the emergent property that this rung contributes to the ladder. Everything above converges on it.

Theorem 10.1 (Decomposition Law of Observables, [S] as mathematics / [H] as physical reading). *Let F be a number field, $\mathcal{H} = \mathcal{H}(F)$ the Hopf algebra of motivic periods, and $\mathcal{L}_G(F) = \overline{\mathcal{H}}/\overline{\mathcal{H}}^2$ its Lie coalgebra with cobracket δ and primitives $P = \bigoplus_n P_n$, $P_1 \cong F^\times \otimes \mathbb{Q}$. Then every motivic amplitude object $\mathcal{A}^m \in \mathcal{H}_n$ admits a canonical, finite, weight-graded decomposition:*

- (1) *its coalgebraic anatomy tree (Theorem 6.4) is finite of depth $\leq n$ (Theorem 6.5);*
- (2) *its leaves are irreducible primitives: the weight-one logarithms $\log x$, $x \in F^\times$ (Theorem 6.6(b)), together with the irreducible higher-weight constants of $\bigoplus_{k \geq 2} P_k$ (such as the odd zeta values), each with vanishing cobracket (Theorem 6.6(c), Theorem 6.7);*
- (3) *the whole object is cogenerated from these primitives by iterated cobrackets (Theorem 6.6(a),(c));*
- (4) *the geometric realizations transport the decomposition functorially (Theorem 6.1), while the numerical period realization is the comodule coaction (Theorem 6.2) whose first slot computes the physical discontinuity/cut (Theorem 7.4),*

subject to the tame hypothesis that $\mathcal{A}^m$ lies in the mixed Tate Hopf algebra; when a non-Tate subquotient appears the law holds only relative to the enlarged elliptic-polylogarithm coalgebra (Theorem 8.1).

Proof. Claim (1) is Theorem 6.5. Claim (2) is Theorem 6.6(b),(c) together with Theorem 6.7. Claim (3) is Theorem 6.6(a),(c). Claim (4) combines Theorem 6.1, Theorem 6.2, and Theorem 7.4. The final tame proviso is Theorem 8.1. $\square$

In the language of the dictionary: *an observable is not an atom; it is a word in a graded Lie coalgebra, spelled from irreducible primitives — logarithmic letters in weight one and irreducible constants in higher weight — and the cobracket is the grammar that governs how the word is read as a sequence of cuts.* This is the emergent property of Rung IV.

$$\begin{array}{ccccc}
\mathcal{A}^m & \xrightarrow{\Delta'} & \mathcal{L} \otimes \mathcal{L} & \xrightarrow{\text{iterate}} & \bigwedge P \\
\text{per} \downarrow & & \downarrow \text{per} \otimes \text{per} & & \downarrow \text{per} \\
\mathcal{A} & \xrightarrow{\text{Disc}} & (\text{cut data}) & \xrightarrow{\text{iterate}} & (\text{logs \& constants})
\end{array}$$

The top row is the coalgebraic anatomy; the bottom row is its numerical shadow; the vertical arrows are the period realization. The diagram commutes by Theorem 6.1 and Theorem 7.4: *to decompose the observable is to decompose the motive and then measure.*

11 Discussion

11.1 Position in the ladder

Rung IV supplies the *operation* that the neighbouring rungs presuppose. Rung III (motives) hands over the actually-constructed Hopf algebra $\mathcal{H}(F)$ on which Δ and δ act; without it, the coproduct would be a formal operation on an unconstructed object. Rung V (regulators) picks out the single realization $\alpha = \text{per}$ from the family in Theorem 6.1, and the well-definedness of its Bloch–Wigner regulator is a shadow of the cobracket kernel of Theorem 7.2. Rung II’s symbol is the maximal iteration of the reduced coproduct (Definition 5.3); Rung I’s raw ratios $F^\times$ are literally the weight-one primitives P_1 (Theorem 6.6(b)). The core object thus touches every other rung.

11.2 Limitations

1. Not every mathematical analogy is a physical theory. Our strongest results (Theorems 4.2, 6.1 and 6.6) are status **S** as mathematics; their universal physical reading is status **H** and is flagged as such at each occurrence.
2. Motives are not literally “the functions of” an object; the functional-essence reading is status **H** throughout. We never present it as Grothendieck’s definition.
3. There is no single universal functor $\text{Math} \rightarrow \text{Phys}$; each realization r_α is a local, domain-indexed channel.
4. Not every physical observable is a period expressible in the tame mixed Tate coalgebra: Theorem 8.1 makes this a precise, citable limitation.
5. The cosmic Galois group (Theorem 9.1) is the most speculative component; it retains status **[H/P]** and, by Theorem 2.1, cannot be upgraded by composition.

11.3 Epistemic status statistics

For transparency (following Section 2.2) we tabulate the status labels attached to the numbered results of this paper.

Status	Count	Results
S (standard)	12	Theorem 3.5, 3.6, 4.2, 5.2, 6.5, 6.6, 6.7, 6.2, 7.2, 7.4, 8.1, 9.2
S/H (std. math, heur. physics)	2	Theorem 7.5, Theorem 10.1
H (heuristic)	2	Theorem 6.3, Theorem 6.8
H/P (heur.speculative)	1	Theorem 9.1

The core structural spine of the paper is entirely status **S**; the physical readings are quarantined as **H**, and the single most speculative claim (cosmic Galois) is **H/P**.

11.4 Open problems

1. A rigorous general proof (or disproof) that the positive-geometry residue tree of Rung I / cluster-coordinate combinatorics is isomorphic to the motivic coaction tree of this rung. This is arguably the single most interesting open problem surfaced by the library.
2. A precise general definition of the cosmic Galois group G_c for an arbitrary QFT (Brown’s conjecture, Theorem 9.1); track the recent K -theoretic reformulations of the Goncharov Lie coalgebra [14, 15] for any bearing on it.
3. A dedicated comparison of the informal “cut/discontinuity” operations of the amplitudes literature (Theorem 7.4) against the formal Beilinson/Borel regulator machinery of Rung V.
4. Extension of the cogeneration theorem (Theorem 6.6) to the elliptic and modular non-Tate regime obstructed in Theorem 8.1: what replaces $P_1 \cong F^\times \otimes \mathbb{Q}$ once one passes to the elliptic-polylogarithm coalgebra?

11.5 Recent progress on the core object

Two very recent constructions bear directly on Theorem 6.6. Kupers–Rudenko–Sierra [14] give an independent K -theoretic construction of exactly this rung’s central object, defining the Goncharov Lie coalgebra via the E_∞ -homology of general linear groups $GL(F)$, determining its structure using Steinberg modules, computing the cobracket, and constructing motivic and Hodge realizations, thereby expressing $K_4^{(3)}(F)$ and the indecomposable part of $K_5^{(3)}(F)$ in terms of Goncharov’s weight-three polylogarithmic complex. Greenberg–Kaufman–Li–Zickert [15] give a complementary explicit construction directly from multiple polylogarithms. Both strengthen the status-**S** footing of the cogeneration theorem.

12 Conclusion

We have constructed the Goncharov Lie coalgebra $\mathcal{L}_G(F)$ as the space of indecomposables of a connected graded Hopf algebra of motivic periods, proved from first principles that its cobracket is well-defined and satisfies co-Jacobi (Theorem 4.2), and established the cogeneration theorem (Theorem 6.6) identifying the weight-one primitives with $F^\times \otimes \mathbb{Q}$ and

showing that the primitives of *all* weights — the logarithms together with the irreducible higher-weight constants (Theorem 6.7) — cogenerate the whole coalgebra. We transported the coaction through the geometric Tannakian realizations (Theorem 6.1), identified the numerical period realization as Brown’s comodule coaction (Theorem 6.2), computed the one-loop cut as its first coaction slot (Theorem 7.4), identified the five-term relation as a co-bracket kernel (Theorem 7.2), and delimited the tame regime with the Non-Tate Obstruction (Theorem 8.1). The synthesis is the *decomposition law of observables* (Theorem 10.1): an observable is a word in a graded Lie coalgebra, spelled from irreducible primitives — logarithmic letters in weight one and irreducible constants in higher weight — and the cobracket is the grammar of its cuts.

The core thesis of the library is thereby made precise on this rung: *a Goncharov-style Lie coalgebra is the bookkeeping system for how physical observables decompose into irreducible informational constituents*. What remains — the passage from this pre-numerical anatomy to an actual measured real number — is the task of Rung V; and the number it returns is itself a new element of Rung I’s domain $F^\times$, closing the ladder into a loop.

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