

Kinematic Domains and Observable Ratios: Cross-Ratios and the Field of Physical Values

Matthew Long

The YonedaAI Collaboration

YonedaAI Research Collective

Chicago, IL

matthew@yonedaai.com · <https://yonedaai.com>

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Abstract

We open a modular research library whose governing thesis is that *a Goncharov-style Lie coalgebra is a bookkeeping system for how physical observables decompose into irreducible informational constituents*. This paper is Rung I of the ladder $I \rightarrow II \rightarrow III \rightarrow IV \rightarrow V \rightarrow \text{Synthesis}$; it constructs the ground floor on which the entire apparatus rests. We treat a field $\mathbb{F}$ as the *kinematic (coordinate) domain* of a physical process, its multiplicative group $\mathbb{F}^\times$ as the group of nonzero scales, ratios, and dimensionless observable parameters, and the cross-ratio of four points on $\mathbb{P}^1(\mathbb{F})$ as the paradigm of a PGL_2 -invariant — and hence frame-independent — observable. We prove that the cross-ratio is a complete invariant of an ordered four-tuple of distinct points (Möbius invariance, Theorem 4.2) and that it identifies the four-point kinematic moduli space $\mathcal{M}_{0,4}$ with the thrice-punctured projective line $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (Theorem 4.8). This last fact is the first concrete hand-off of the ladder: the “domain of observable values” for four-point kinematics is *exactly* the analytic domain on which the classical polylogarithm Li_n of Rung II is single-valued away from its branch points. We record, as a forward pointer, the motivic fact that $\mathbb{F}^\times \otimes \mathbb{Q}$ is canonically the group of weight-one primitives $\text{Ext}_{\text{MTM}(\mathbb{F})}^1(\mathbb{Q}(0), \mathbb{Q}(1))$ of the coalgebra built in Rung IV, so that Rung I’s raw ratios are literally the irreducible weight-one building blocks of the whole hierarchy. Throughout, we apply a math $\leftrightarrow$ physics dictionary and a three-valued epistemic status system (S: standard mathematics; H: strong heuristic representation; P: speculative ontology) verbatim to every translation entry, so that the reader can see at a glance which claims are theorems and which are interpretive scaffolding. The emergent property established at this rung is **a disciplined domain of observable values**: a gauge-reduced arena of ratios, scales, and dimensionless invariants on which every later construction of the library is built.

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1 Introduction

1.1 The governing perspective

A physical observable — a scattering amplitude, a Feynman integral, a period, an entropy — is here treated as the *numerical realization* of a deeper, pre-numerical, structured object. The question “what is this observable made of?” is answered not by further computation but by a decomposition law: a coproduct or cobracket that exposes the observable’s primitive, irreducible constituents (its cuts, discontinuities, and factorization channels). The organizing slogans of the library, which recur in every paper of the series, are:

observable = Real(motivic information)
coalgebra = grammar of physical decomposition
Physics observes numerical shadows of deeper motivic structures.
Physical reality is the realization layer of structured mathematical information.

The thesis in one sentence: *a Goncharov-style Lie coalgebra is not merely a formal algebraic object; it is a bookkeeping system for how physical quantities break apart into irreducible informational constituents*. Making that sentence precise, and proving pieces of it, is the work of the whole library. The present paper builds the arena in which the observables to be decomposed actually live.

1.2 A modular, not unified, framework

This library is deliberately **modular**. It is not a single grand unified construction but a ladder of standalone, individually defensible papers, each of which composes with its neighbour to add exactly one emergent structural property that the level below cannot express on its own. Table 1 records the five rungs and the synthesis that recomposes them.

1.3 This paper’s place in the ladder

Rung I is the ground floor. Before one can ask how an amplitude decomposes, one must say precisely *what an amplitude is a function of*: what values its arguments may take, what redundancies (frame or gauge choices) those arguments carry, and which combinations of them are honest, frame-independent observables. We answer this by identifying:

- the *field* $\mathbb{F}$ as the domain of allowed values of a physical parameter;

Rung	New structure	Emergent property
I	Field $\mathbb{F}$, group $\mathbb{F}^\times$, cross-ratios, moduli $\text{Conf}_n(\mathbb{P}^1) = \mathcal{M}_{0,n}$	<i>A disciplined domain of observable values</i> — the raw arena of ratios, scales, and dimensionless invariants
II	$\text{Li}_n(x)$, iterated integrals, weight/depth grading, the symbol	<i>Transcendental depth = loop order</i> — special functions whose complexity tracks physical complexity
III	Motives $\text{Mot}(X)$, realizations, comparison isomorphism, mixed Tate motives	<i>The pre-numerical object behind measurement</i> — the transcendental numbers of Rung II are periods
IV	Hopf algebra of motivic periods, coproduct Δ , cobracket δ , $\mathcal{L}_G(\mathbb{F})$	<i>The decomposition law of observables</i> — coalgebraic anatomy terminating at weight-one primitives
V	Regulator maps, Deligne–Beilinson cohomology, period pairing $\int_\Gamma \omega$	<i>Passage from motive to measurable real number</i>
Synth.	Recomposition into observable = per(motive)	<i>Measurement as the closure of the ladder</i>

Table 1: The modular ladder. Each rung is standalone; composition creates the emergent property in the right-hand column. This paper is Rung I.

- the *multiplicative group* $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ as the group of nonzero scales and ratios;
- the *cross-ratio* as the universal PGL_2 -invariant of four marked points, and hence the model of a dimensionless, scale-free observable;
- the *moduli space* $\text{Conf}_n(\mathbb{P}^1) = \mathcal{M}_{0,n}$ as the space of physically distinct n -particle kinematic configurations.

The emergent property we establish is a **disciplined domain of observable values**: “domain” because it is where the arguments of every later function live; “disciplined” because the PGL_2 -quotient removes the labelling redundancy and leaves exactly the invariant ratios, and because — as Theorem 4.8 shows — the resulting arena is *precisely* the analytic domain the next rung requires.

1.4 Standalone-library discipline

This project is self-contained. All citations are to the external mathematical and physical literature listed in the references; no sibling project is cited, linked, or relied upon. The reader needs only the standard background in field theory, projective geometry, and scattering amplitudes recalled in Section 3.

1.5 Outline

Section 2 fixes the math $\leftrightarrow$ physics dictionary, the S/H/P epistemic status system, and the master formulas that recur across the library. Section 3 recalls fields, multiplicative groups,

Mathematics	Physical representation
Field $\mathbb{F}$	Kinematic/coordinate domain of possible values
Element of $\mathbb{F}^\times$	Ratio, cross-ratio, scale, observable parameter
Polylogarithm $\text{Li}_n(x)$	Iterated propagation / layered amplitude contribution
Period	Measurable physical number obtained from geometry
Motive	Pre-numerical informational object behind the measurement
Lie algebra	Infinitesimal symmetry / generator structure
Lie coalgebra	Decomposition law of observables
Cobracket δ	Factorization channel, cut, boundary, or information split
Weight grading	Complexity depth / loop order / transcendental depth
Primitive element	Irreducible informational unit
Coproduct	Full hierarchical decomposition
Regulator map	Passage from motivic object to physical real number
Hodge/de Rham realization	Analytic form seen by calculation
Betti realization	Topological/path/integration-cycle form
Period pairing	Physical observable as pairing of form and cycle

Table 2: The seed math $\leftrightarrow$ physics dictionary, reproduced verbatim in every paper of the library.

the projective line, and the Möbius action. Section 4 proves Möbius invariance of the cross-ratio and the trivialization of $\mathcal{M}_{0,4}$. Section 5 treats configuration spaces and their boundary structure. Section 6 records the motivic role of $\mathbb{F}^\times$ as weight-one classifying data. Section 7 gives the topic-specific dictionary. Section 8 works the four-point cross-ratio as an amplitude’s domain, and Section 9 treats higher-point domains via cluster coordinates, the positive Grassmannian, and the scattering equations. Sections 10 to 12 collect results, discuss limitations and open problems, and conclude.

2 The Dictionary, the Status System, and the Master Formulas

2.1 The math $\leftrightarrow$ physics dictionary

Every paper in this library reproduces, verbatim, the seed translation table of Table 2. It is the interface between the mathematical objects (left column) and their proposed physical readings (right column). Later sections refine it with topic-specific entries; those refinements never replace the seed table.

2.2 The epistemic status system (S/H/P)

The dictionary mixes rigorous mathematics with interpretive proposals of varying boldness. To keep the two apart we tag *every* dictionary entry and *every* claim with one of three status labels, following Table 3.

Composite labels **S/H** and **H/P** occur where an entry is standard as pure mathematics but heuristic or speculative in its physical reading. We preserve composite labels wherever

Label	Meaning	Canonical example
S	Standard use in mathematics or mathematical physics	The Goncharov coproduct on motivic polylogarithms
H	Strong heuristic representation-dictionary entry	Motive as “functional essence”; cobracket as “factorization channel”
P	Speculative philosophical ontology	Physical reality as motivic-categorical realization; a universal cosmic Galois group acting on all amplitudes

Table 3: The three-valued epistemic status system. Composite labels S/H and H/P occur where an entry is standard as pure mathematics but heuristic or speculative in its physical reading.

they occur.

Remark 2.1 (Status monotonicity). If two representation entries or theorems compose, the status of the composite is the minimum under the order $S < H < P$: a chain of translations is only as reliable as its weakest link. A **P**-labelled step cannot be upgraded to **S** merely by composing it with otherwise-rigorous results. This rule is used silently throughout the library and explicitly in the synthesis paper’s epistemic audit.

2.3 Master formulas

The following notation is fixed for the whole library; only the first two lines are used substantively in this paper, but all are recorded so the reader can see where Rung I feeds:

$$\text{observable} = \text{Real}(\text{abstract structure}) \quad (\text{compressed master slogan}) \quad (1)$$

$$\mathcal{A} = \text{per}(\mathcal{A}^{\mathfrak{m}}) \quad (\text{amplitude} = \text{period of motivic amplitude}) \quad (2)$$

$$\Delta(\mathcal{A}^{\mathfrak{m}}) = \sum_i \mathcal{A}_{i,1}^{\mathfrak{m}} \otimes \mathcal{A}_{i,2}^{\mathfrak{m}} \quad (\text{coproduct / full decomposition}) \quad (3)$$

$$\delta : \mathcal{L}_G(\mathbb{F}) \rightarrow \bigwedge^2 \mathcal{L}_G(\mathbb{F}) \quad (\text{cobracket / primitive split}) \quad (4)$$

$$\text{per}(\Pi) = \int_{\Gamma} \omega, \quad \Pi = (X, D, \omega, \Gamma) \quad (\text{period pairing}) \quad (5)$$

Here $\mathbb{F}$ is a field (typically a number field, or a field of kinematic invariants), $\mathbb{F}^{\times}$ its multiplicative group; $\mathcal{A}^{\mathfrak{m}}$ is a pre-numerical element of a Hopf algebra $\mathcal{H}$ of motivic periods; and $\mathcal{L}_G(\mathbb{F}) = \bigoplus_{n \geq 1} \mathcal{L}_G(\mathbb{F})_n$ is the Goncharov Lie coalgebra of primitives, graded by weight n . Rung I is responsible for the objects $\mathbb{F}$ and $\mathbb{F}^{\times}$ appearing in (2) and, as Section 6 explains, for the weight-one piece $\mathcal{L}_G(\mathbb{F})_1$ of (4).

3 Mathematical Framework: Fields, Ratios, and the Projective Line

3.1 Fields as kinematic domains

Definition 3.1 (Kinematic domain, status **S/H**). For a physical process with n particles, the *kinematic domain* is the field (or function field) $\mathbb{F}$ generated by the independent Lorentz-

invariant combinations of the momenta — masses, Mandelstam invariants $s_{ij} = (p_i + p_j)^2$, momentum-twistor brackets — subject to momentum conservation and the on-shell constraints. The translation “physical parameter $\mapsto$ element of $\mathbb{F}$ ” is status **S/H**: the algebra of invariants is a standard object (**S**); its reading as “the domain of everything the process could be a function of” is a heuristic framing (**H**).

Concretely, $\mathbb{F}$ ranges over $\mathbb{Q}$, a number field, $\mathbb{R}$, $\mathbb{C}$, or a field of rational functions of kinematic invariants, depending on the arithmetic or analytic question at hand. The choice of $\mathbb{F}$ is itself physical data: working over $\mathbb{Q}$ isolates the arithmetic content (which periods and multiple zeta values can appear), while working over a function field isolates the analytic dependence on kinematics.

3.2 The multiplicative group as observable ratios

Definition 3.2 (The scale group $\mathbb{F}^\times$, status **S/H**). The multiplicative group $\mathbb{F}^\times = \mathbb{F} \setminus \{0\}$ is the group of nonzero scales, ratios, and dimensionless observable parameters. An element $x \in \mathbb{F}^\times$ is read as an energy ratio, a mass ratio, a coupling ratio, or — most importantly for this paper — a cross-ratio.

The distinction between $\mathbb{F}$ and $\mathbb{F}^\times$ is not pedantry. Physics is largely a science of *ratios*: a bare scale carries units and depends on a choice of reference, but a ratio of two like quantities is a pure number, comparable across experiments. The passage from $\mathbb{F}$ to $\mathbb{F}^\times$ is the first act of the “discipline” promised in the title. In Section 6 we shall see that $\mathbb{F}^\times \otimes \mathbb{Q}$ is not merely a convenient group but is *exactly* the space of irreducible weight-one informational units of the entire library.

3.3 The projective line and the Möbius group

Let $\mathbb{P}^1(\mathbb{F}) = \mathbb{F} \cup \{\infty\}$ be the projective line over $\mathbb{F}$, i.e. the set of $\mathbb{F}$ -points of $\mathbb{P}^1$, with homogeneous coordinates $[x_0 : x_1]$ and affine chart $x = x_1/x_0$. The group $\mathrm{GL}_2(\mathbb{F})$ acts on $\mathbb{P}^1(\mathbb{F})$ through its quotient $\mathrm{PGL}_2(\mathbb{F}) = \mathrm{GL}_2(\mathbb{F})/\mathbb{F}^\times$ by *Möbius (fractional linear) transformations*

$$g \cdot x = \frac{ax + b}{cx + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc \neq 0, \quad (6)$$

with the usual conventions $g \cdot \infty = a/c$ and $g \cdot (-d/c) = \infty$.

Lemma 3.3 (Simple transitivity on triples, status **S**). $\mathrm{PGL}_2(\mathbb{F})$ acts simply transitively on ordered triples of distinct points of $\mathbb{P}^1(\mathbb{F})$. Equivalently: given two ordered triples of distinct points there is a unique Möbius transformation carrying the first to the second.

Proof. It suffices to show that for any ordered triple (a, b, c) of distinct points there is a unique $g \in \mathrm{PGL}_2(\mathbb{F})$ with $g(a) = 0$, $g(b) = 1$, $g(c) = \infty$; composing one such map with the inverse of another then carries any triple to any other, uniquely. Existence: the map

$$h(x) = \frac{(x - a)(b - c)}{(x - c)(b - a)} \quad (7)$$

is Möbius (its matrix has nonzero determinant because a, b, c are distinct) and satisfies $h(a) = 0$, $h(b) = 1$, $h(c) = \infty$ by direct substitution. Uniqueness: if g_1, g_2 both carry (a, b, c) to $(0, 1, \infty)$, then $g_2 g_1^{-1}$ fixes $0, 1, \infty$; a Möbius transformation fixing three distinct points is the identity, because $g \cdot x = x$ becomes $cx^2 + (d - a)x - b = 0$, a nonzero polynomial of degree ≤ 2 unless $b = c = 0$ and $a = d$, i.e. unless g is the identity in PGL_2 . Hence $g_1 = g_2$. $\square$

Theorem 3.3 is the algebraic engine behind every statement of this paper: three points can always be sent to the canonical triple $(0, 1, \infty)$, and the fourth point's image is then a complete invariant. That invariant is the cross-ratio.

4 Cross-Ratios and Möbius Invariance

4.1 The cross-ratio

Definition 4.1 (Cross-ratio, status **S**). The *cross-ratio* of an ordered four-tuple (x_1, x_2, x_3, x_4) of points of $\mathbb{P}^1(\mathbb{F})$, with x_1, x_2, x_3, x_4 pairwise distinct, is

$$r(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)} \in \mathbb{F}^\times, \quad (8)$$

with the standard conventions for the value at ∞ obtained by cancelling the two factors containing ∞ .

The cross-ratio lands in $\mathbb{F}^\times$ precisely because the four points are distinct: numerator and denominator are then both nonzero. It is the physical archetype of a *dimensionless* observable — it is unchanged under a common rescaling $x_i \mapsto \lambda x_i$ and, more strongly, under the full Möbius group.

4.2 Möbius invariance

Proposition 4.2 (Möbius invariance of the cross-ratio, status **S**). *For every $g \in \mathrm{PGL}_2(\mathbb{F})$ and every four-tuple of distinct points,*

$$r(gx_1, gx_2, gx_3, gx_4) = r(x_1, x_2, x_3, x_4).$$

Proof. Write $g \cdot x = (ax + b)/(cx + d)$ with $ad - bc \neq 0$. A direct computation gives, for all i, j ,

$$gx_i - gx_j = \frac{ax_i + b}{cx_i + d} - \frac{ax_j + b}{cx_j + d} = \frac{(ad - bc)(x_i - x_j)}{(cx_i + d)(cx_j + d)}. \quad (9)$$

Substitute (9) into the four differences of the cross-ratio (8). Each factor $(cx_i + d)$ appears once in a numerator difference (through $gx_i - gx_j$) and once in a denominator difference (through $gx_i - gx_k$), because each index $i \in \{1, 2, 3, 4\}$ occurs exactly once in the two numerator factors and exactly once in the two denominator factors of (8). Hence all four factors $(cx_i + d)$ cancel. The factor $(ad - bc)$ appears twice in the numerator and twice in the denominator, so the two copies cancel as well. What remains is exactly $r(x_1, x_2, x_3, x_4)$. $\square$

Remark 4.3 (Physical reading, status **S/H**). In the amplitudes literature, Möbius (more precisely, dual-conformal) invariance is the statement that a physical observable does not depend on the redundant labelling of an on-shell kinematic configuration. Theorem 4.2 is the exact mathematical content of that physical principle for four marked points: the cross-ratio is the honest, frame-independent datum, while the individual coordinates x_i carry gauge redundancy. Status **S** as mathematics; the identification “ PGL_2 redundancy = physical frame choice” is the **H** overlay.

4.3 The anharmonic group

The cross-ratio depends on the *ordering* of the four points. Permuting the four arguments produces at most six distinct values, generated by $r \mapsto 1/r$ and $r \mapsto 1 - r$:

$$\left\{ r, \frac{1}{r}, 1 - r, \frac{1}{1-r}, \frac{r}{r-1}, \frac{r-1}{r} \right\}. \quad (10)$$

Lemma 4.4 (The anharmonic action, status **S**). *The symmetric group S_4 acts on the cross-ratio of an ordered four-tuple through its quotient $S_4/V_4 \cong S_3$, where V_4 is the Klein four-group; the resulting S_3 -action on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is generated by the involutions $r \mapsto 1/r$ and $r \mapsto 1 - r$ and has orbit (10).*

Proof. The Klein four-group $V_4 \subset S_4$ of double transpositions fixes the cross-ratio: e.g. the swap $(x_1x_2)(x_3x_4)$ replaces (8) by $\frac{(x_2-x_4)(x_1-x_3)}{(x_2-x_3)(x_1-x_4)} = r$, and similarly for the other two nontrivial elements. Hence the action factors through $S_4/V_4 \cong S_3$. The transposition (x_3x_4) sends $r \mapsto 1/r$ and (x_2x_3) sends $r \mapsto 1 - r$; these two involutions generate a dihedral group of order six acting as (10), and one checks the six values are distinct for generic r . $\square$

The two exceptional orbits are physically and arithmetically meaningful. The *harmonic* values $r \in \{-1, 2, \frac{1}{2}\}$ form an orbit of size three (each fixed by one of the three transpositions), and the *equianharmonic* values — the two primitive sixth roots of unity $r = e^{\pm i\pi/3}$, i.e. the roots of $r^2 - r + 1 = 0$, each fixed by the order-three element $r \mapsto 1/(1 - r)$ — form an orbit of size two. These are the fixed configurations of the anharmonic symmetry and are exactly the points where the associated elliptic curve $y^2 = x(x - 1)(x - r)$ acquires extra automorphisms — a foreshadowing of the non-Tate boundary that the library encounters at Rung III/IV.

4.4 The cross-ratio generates the field of invariants

Möbius invariance (Theorem 4.2) shows the cross-ratio is *an* invariant; the following shows it is essentially the *only* one, so that “dimensionless four-point observable” and “function of the cross-ratio” are synonymous.

Proposition 4.5 (Completeness of the cross-ratio, status **S**). *Let $\mathbb{F}$ be infinite. The field of $\mathrm{PGL}_2(\mathbb{F})$ -invariant rational functions of four ordered distinct points of $\mathbb{P}^1$ is the rational function field $\mathbb{F}(r(x_1, x_2, x_3, x_4))$ generated by the single cross-ratio. Equivalently, a rational function $\varphi(x_1, x_2, x_3, x_4)$ is Möbius-invariant if and only if $\varphi = \psi \circ r$ for a rational function ψ of one variable.*

Proof. The “if” direction is Theorem 4.2. Conversely, the configuration space $\text{Conf}_4(\mathbb{P}^1)$ has dimension $4 - 3 = 1$ and, by Theorem 4.8, is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with coordinate r . A PGL_2 -invariant rational function descends to a rational function on this one-dimensional quotient, i.e. to a rational function of r ; and every rational function of r pulls back to an invariant. Hence the invariant field is $\mathbb{F}(r)$. (Concretely: use Theorem 3.3 to fix $(x_2, x_3, x_4) = (1, 0, \infty)$, so that an invariant φ is determined by its restriction to the slice, a rational function of the single remaining coordinate $x_1 = r$.) $\square$

Example 4.6 (A numerical cross-ratio and its anharmonic orbit, status **S**). Take $\mathbb{F} = \mathbb{Q}$ and the four points $(x_1, x_2, x_3, x_4) = (3, 1, 0, \infty) \in \mathbb{P}^1(\mathbb{Q})$. Cancelling the two factors containing ∞ in (8) gives

$$r(3, 1, 0, \infty) = \frac{(3 - 0)}{(1 - 0)} = 3.$$

Its anharmonic orbit (10) is

$$\left\{ 3, \frac{1}{3}, -2, -\frac{1}{2}, \frac{3}{2}, \frac{2}{3} \right\},$$

the six values obtained by reordering the four points; all lie in $\mathbb{Q}^\times \setminus \{1\}$, as Theorem 4.8 requires. Each is the image of one of the six orderings under a single Möbius normalization, and any $\text{PGL}_2(\mathbb{Q})$ -invariant rational function of the four points is, by Theorem 4.5, a rational function of the single value 3.

4.5 The four-point moduli space

Definition 4.7 (Configuration space, status **S**). For $n \geq 3$ let

$$\text{Conf}_n(\mathbb{P}^1) := \{(x_1, \dots, x_n) \in (\mathbb{P}^1)^n : x_i \neq x_j \text{ for } i \neq j\} / \text{PGL}_2,$$

the moduli space of n ordered distinct marked points on the projective line modulo the diagonal Möbius action. It is smooth of dimension $n - 3$ and coincides with the moduli space $\mathcal{M}_{0,n}$ of genus-zero curves with n marked points.

Theorem 4.8 (The cross-ratio trivializes $\mathcal{M}_{0,4}$, status **S**). *The map*

$$z : \text{Conf}_4(\mathbb{P}^1) \longrightarrow \mathbb{P}^1 \setminus \{0, 1, \infty\}, \quad (x_1, x_2, x_3, x_4) \longmapsto r(x_1, x_2, x_3, x_4),$$

is an isomorphism. Equivalently, every ordered four-tuple of distinct points is PGL_2 -equivalent to $(\lambda, 1, 0, \infty)$ for a unique $\lambda \in \mathbb{F} \setminus \{0, 1\}$, and $\lambda = r(x_1, x_2, x_3, x_4)$.

Proof. By Theorem 3.3 there is a unique $g \in \text{PGL}_2(\mathbb{F})$ with $g(x_2) = 1$, $g(x_3) = 0$, $g(x_4) = \infty$; explicitly g is the map (7) with $(a, b, c) = (x_3, x_2, x_4)$ in the roles of (a, b, c) there, i.e.

$$g(x) = \frac{(x - x_3)(x_2 - x_4)}{(x - x_4)(x_2 - x_3)}.$$

Direct substitution gives $g(x_2) = 1$, $g(x_3) = 0$, $g(x_4) = \infty$ and

$$g(x_1) = \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)} = r(x_1, x_2, x_3, x_4) =: \lambda.$$

$$\begin{array}{ccc}
\text{Conf}_4(\mathbb{P}^1) = \mathcal{M}_{0,4} & \xrightarrow[\sim]{z = \text{cross-ratio}} & \mathbb{P}^1 \setminus \{0, 1, \infty\} \\
\uparrow \text{gauge-fix PGL}_2 & & \downarrow \text{domain of Li}_n \\
\text{4-point kinematics} & & \text{Rung II: Li}_n(z), \text{ iterated integrals}
\end{array}$$

Figure 1: The cross-ratio identifies the gauge-reduced four-point kinematic moduli space with the thrice-punctured line, which is exactly the domain handed to Rung II.

Thus $(x_1, x_2, x_3, x_4) \sim (\lambda, 1, 0, \infty)$. By Theorem 4.2 the value λ is a PGL_2 -invariant, hence descends to $\text{Conf}_4(\mathbb{P}^1)$; and it determines the configuration uniquely, since the normalized representative $(\lambda, 1, 0, \infty)$ is recovered from λ . Finally $\lambda \neq 0, 1, \infty$ exactly when the four points remain distinct: one computes directly from (8) that $\lambda = \infty \iff x_1 = x_4$ or $x_2 = x_3$; $\lambda = 0 \iff x_1 = x_3$ or $x_2 = x_4$; and $\lambda = 1 \iff x_1 = x_2$ or $x_3 = x_4$ (the last from $(x_1 - x_3)(x_2 - x_4) = (x_1 - x_4)(x_2 - x_3)$, which simplifies to $(x_1 - x_2)(x_3 - x_4) = 0$). Therefore z is a bijection onto $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and it is an isomorphism of varieties because it and its inverse (normalize, read off λ) are given by rational maps. $\square$

Remark 4.9 (The first hand-off of the ladder, status **S/H**). Theorem 4.8 says the “domain of observable values” for a four-point kinematic configuration is *exactly* $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. This is precisely the domain on which the classical polylogarithm $\text{Li}_n(z)$ and its relatives are single-valued and holomorphic away from branch points at $0, 1, \infty$. Rung I therefore does not merely supply “some field”; it supplies the *specific punctured domain* that Rung II’s special functions are functions of. The three punctures are not arbitrary: they are the three ways two of the four marked points can collide, i.e. the three boundary components of the compactified moduli space, and — physically — the three degeneration channels of the configuration. Status **S** as the geometric statement; the reading of the punctures as physical thresholds is **S/H**.

5 Configuration Spaces and the Geometry of Marked Points

5.1 Coordinates on $\mathcal{M}_{0,n}$

For $n > 4$ the moduli space $\mathcal{M}_{0,n}$ has dimension $n - 3 > 1$, and a single cross-ratio no longer coordinatizes it. Instead its coordinate ring is generated by cross-ratios of four-element subsets of the n marked points. Fixing three of the points to $(0, 1, \infty)$ via Theorem 3.3 leaves $n - 3$ free points, and the $n - 3$ cross-ratios

$$u_i = r(x_i, x_1, x_2, x_3), \quad i = 4, \dots, n, \quad (11)$$

give a system of coordinates realizing $\mathcal{M}_{0,n}$ as an open subvariety of $(\mathbb{P}^1 \setminus \{0, 1, \infty\})^{n-3}$ cut out by the conditions that the free points remain distinct from one another. These are the *dihedral* or *simplicial* coordinates familiar from the moduli-space description of tree-level string and field-theory amplitudes.

5.2 Compactification and boundary divisors

The Deligne–Mumford compactification $\overline{\mathcal{M}}_{0,n}$ adds a normal-crossings boundary divisor whose components are indexed by the ways the marked points can collide into stable nodal configurations. Each boundary component corresponds to a partition of the marked points into two clusters of size ≥ 2 — geometrically, a bubbling-off of a component of the curve.

Remark 5.1 (Boundaries as factorization channels, status **S/H**). Physically, a boundary divisor of $\overline{\mathcal{M}}_{0,n}$ is a *factorization channel*: the degeneration in which a subset of external legs becomes collinear/on-shell, so that the amplitude factorizes into a product of lower-point amplitudes glued along an internal line. This is the geometric origin of the dictionary entry “cobracket $\delta \leftrightarrow$ factorization channel, cut, boundary, or information split.” At Rung I this correspondence is purely geometric (**S**); its promotion to a statement about the coalgebra’s cobracket is Rung IV’s task, and its universality across all amplitudes is heuristic (**H**). We flag here, as a forward pointer, the conjecture that the tree of boundary/residue degenerations of a positive geometry matches the tree of iterated coactions of Rung IV — one of the sharpest open problems the library raises.

Example 5.2 (The pentagon $\mathcal{M}_{0,5}$, status **S**). The space $\mathcal{M}_{0,5}$ is two-dimensional. Fixing $(x_1, x_2, x_3) = (0, 1, \infty)$ leaves two free points x_4, x_5 , coordinatized by the pair of cross-ratios (u, v) from (11). Its compactification $\overline{\mathcal{M}}_{0,5}$ is a del Pezzo surface of degree five, and one must keep two different objects distinct.

- *The complex algebraic boundary.* The boundary divisor of the complex surface $\overline{\mathcal{M}}_{0,5}$ consists of exactly *ten* irreducible components, one for each of the ten ways to split five marked points into a pair and a triple. Every one of these ten components is a (-1) -curve (each isomorphic to $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$), hence one-dimensional; they meet pairwise in fifteen points, the maximally degenerate (doubly nodal) configurations. There are no “vertices” in the algebraic boundary — only curves and their 15 intersection points.
- *The real positive geometry.* The totally-positive part $\overline{\mathcal{M}}_{0,5}(\mathbb{R})_{\geq 0}$ is a distinct, and combinatorially smaller, object: the Stasheff associahedron K_4 , i.e. a pentagon. Its boundary has *five* edges (dimension one), one for each cyclically adjacent pair-partition, and *five* vertices (dimension zero), one for each maximal degeneration where two adjacent partitions coincide. The pentagon is the real positive shadow of the ten-curve complex boundary; a one-dimensional boundary curve of the complex surface is not the same thing as a zero-dimensional vertex of the real pentagon.

Either way, the boundary combinatorics carry exactly the five-term relation for the dilogarithm, which reappears at Rung II as the fundamental functional equation of Li_2 and at Rung IV as the defining relation of the Bloch group. Thus even the ground-floor geometry of Rung I already encodes the connective tissue of the whole library.

6 $\mathbb{F}^\times$ as Weight-One Motivic Classifying Data

We now record the precise sense in which Rung I’s raw ratios $\mathbb{F}^\times$ are not merely inputs but are the irreducible weight-one building blocks of the entire coalgebraic edifice erected at

$$x \in \mathbb{F}^\times \longmapsto [H^1(\mathbb{G}_m, \{1, x\})] \in \text{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \xrightarrow{\text{per}} \log x \in \mathbb{C}$$

Figure 2: The weight-one hand-off: a ratio $x \in \mathbb{F}^\times$ is the period datum of a Kummer motive, whose period is $\log x$. This is the base case of the entire realization pipeline $\mathcal{A} = \text{per}(\mathcal{A}^m)$.

Rung IV. The following statement is proved in full at Rung III/IV; we quote it here as a forward pointer because it is the arithmetic justification for calling $\mathbb{F}^\times$ the “ground floor.”

Proposition 6.1 ($\mathbb{F}^\times$ as weight-one classifying data, status **S** as mathematics). *There is a canonical isomorphism*

$$\text{Ext}_{\text{MTM}(\mathbb{F})}^1(\mathbb{Q}(0), \mathbb{Q}(1)) \cong \mathbb{F}^\times \otimes \mathbb{Q}, \quad (12)$$

under which the extension class attached to $x \in \mathbb{F}^\times$ has period $\log x$. Consequently the weight-one part of the Goncharov Lie coalgebra satisfies $\mathcal{L}_G(\mathbb{F})_1 \cong \mathbb{F}^\times \otimes \mathbb{Q}$, and every weight-one primitive is the class of a Kummer motive $H^1(\mathbb{G}_m, \{1, x\})$.

We do not reprove (12) here — it belongs to the mixed-Tate machinery of Rung III — but we stress its meaning for the present rung. The multiplicative group $\mathbb{F}^\times$ that Theorem 3.2 introduced as “observable ratios” turns out to be, after tensoring with $\mathbb{Q}$, *literally* the space $\mathcal{L}_G(\mathbb{F})_1$ of irreducible weight-one informational units. Every higher-weight element of the coalgebra is cogenerated, via iterated cobrackets, from these weight-one primitives (this is the structural anchor theorem of Rung IV). In the language of the dictionary:

Remark 6.2 (Status of the “ground floor” reading). The isomorphism (12) is status **S** as pure mathematics. The physical reading — “the domain of observable ratios is the ground floor of the whole informational hierarchy, and all amplitude complexity is generated by iterating single cuts of logarithmic building blocks” — is status **H**. By Theorem 2.1 (status monotonicity), any downstream claim that composes this heuristic with the rigorous (12) inherits status **H**.

7 The Topic-Specific Dictionary

Table 4 refines the seed dictionary of Table 2 for the objects of Rung I, with the semantic meaning and epistemic status of each entry made explicit. It is the translation table a reader should keep in view for the rest of the paper.

8 Worked Example: The Four-Point Cross-Ratio as the Amplitude’s Domain

We now make the abstract construction concrete on the paradigmatic case of a four-point massless amplitude, and then indicate the six-point (hexagon) generalization that produces genuine dual-conformal cross-ratios.

Mathematics	Physical representation	Semantic meaning	Status
Field $\mathbb{F}$	Kinematic or coordinate domain	Allowed values of physical parameters	S/H
$\mathbb{F}^\times$	Nonzero scales or ratios	Energies, masses, cross-ratios, coupling ratios	S/H
Cross-ratio	Conformal invariant	Dimensionless observable independent of scale	S
$\mathrm{PGL}_2(\mathbb{F})$ -action	Frame/gauge choice on 4 marked points	Redundancy in labelling identical kinematics	S
$\mathrm{Conf}_n(\mathbb{P}^1) = \mathcal{M}_{0,n}$	Moduli of n -particle kinematic configurations	Physically distinct on-shell configurations	S
Cluster $\mathcal{X}$ -coordinate	Natural amplitude variable	Symbol letter or physical cross-ratio	S
Positive Grassmannian	Positive cell geometry	On-shell amplitude combinatorics	S
Scattering equations (CHY)	Map from punctured-sphere moduli to kinematics	Realizes $\mathcal{M}_{0,n}$ as the home of massless kinematics	S

Table 4: Topic-specific dictionary for Rung I, with semantic gloss and S/H/P status.

8.1 Massless four-point kinematics

Consider four massless momenta p_1, p_2, p_3, p_4 with $\sum_i p_i = 0$ and $p_i^2 = 0$. The Mandelstam invariants

$$s = (p_1 + p_2)^2, \quad t = (p_2 + p_3)^2, \quad u = (p_1 + p_3)^2$$

satisfy $s + t + u = \sum_i p_i^2 = 0$, so only two are independent, and the single dimensionless ratio

$$x = -\frac{t}{s} \in \mathbb{F}^\times \tag{13}$$

captures all scale-free content of the configuration ($u/s = -1 - t/s$ is then determined). This $x \in \mathbb{F}^\times$ is the four-point instance of Theorem 3.2’s “observable ratio.” It is a genuine element of the multiplicative group of the kinematic field, and it is what any dimensionless four-point observable can depend on.

8.2 Dual-conformal cross-ratios at six points

At four points there is no nontrivial dual-conformal cross-ratio built from the dual (region) coordinates; the first appear at six points. For a six-point configuration in momentum-twistor space, dual conformal symmetry (the PGL -type redundancy of Theorem 4.2, now acting on the dual configuration) leaves three independent cross-ratios

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \quad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \quad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{36}^2 x_{25}^2}, \tag{14}$$

the hexagon cross-ratios of Goncharov–Spradlin–Vergu–Volovich. Each $u_i \in \mathbb{F}^\times$ is a cross-ratio in the exact sense of Theorem 4.1: it is invariant under the dual-conformal group, and the triple (u_1, u_2, u_3) coordinatizes the reduced six-point kinematic configuration up to residual redundancy. The six-point remainder function of planar $\mathcal{N} = 4$ super-Yang–Mills is a transcendental function on exactly this three-dimensional domain of cross-ratios — a Rung II object living on a Rung I domain.

Remark 8.1 (Domain before function, status **S/H**). The point of this example is a separation of concerns that structures the whole library. Rung I fixes the *domain*: the amplitude is a function of the cross-ratios (13), (14), each an element of $\mathbb{F}^\times$, on the moduli space $\mathcal{M}_{0,n}$. Which *function* of that domain the amplitude equals — a polylogarithm, an iterated integral, an elliptic generalization — is Rung II’s question, and how that function *decomposes* is Rung IV’s. Confusing domain with function is the standard source of confusion in the amplitudes literature; the modular ladder keeps them apart by design.

8.3 The cross-ratio in conformal field theory

The same discipline governs correlation functions in conformal field theory, and the parallel is worth drawing because it is the oldest and cleanest instance of “domain before function.” Conformal invariance fixes the two- and three-point functions of primary operators completely; the first place dynamical information can hide is the four-point function. For four scalar primaries $\mathcal{O}_i$ of dimensions Δ_i at positions z_i , conformal (special conformal + translation + rotation + dilatation) invariance forces

$$\langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_4(z_4) \rangle = \frac{\mathcal{G}(u, v)}{\prod_{i < j} z_{ij}^{\Delta_i + \Delta_j - \frac{1}{3}\Delta}}, \quad \Delta = \sum_i \Delta_i, \quad (15)$$

where the entire nontrivial content is packaged in a function $\mathcal{G}$ of the two independent conformal cross-ratios

$$u = \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2}, \quad v = \frac{z_{14}^2 z_{23}^2}{z_{13}^2 z_{24}^2}, \quad z_{ij} = z_i - z_j. \quad (16)$$

Each of u, v is a cross-ratio in the sense of Theorem 4.1, up to the anharmonic reordering of Theorem 4.4 and (in $d > 1$ dimensions) an overall squaring: in a single holomorphic variable u is the square of the anharmonic partner $z_{12}z_{34}/(z_{13}z_{24})$ of $r(z_1, z_2, z_3, z_4)$. The pair (u, v) coordinatizes the reduced configuration exactly as the simplicial coordinates (11) coordinatize $\mathcal{M}_{0,4}$ in each chiral sector. The conformal bootstrap is then the study of which *functions* $\mathcal{G}(u, v)$ are consistent — precisely a Rung II-and-beyond question posed on the Rung I domain of cross-ratios. That the crossing equation of the bootstrap is a relation among the values of $\mathcal{G}$ at the anharmonically-related arguments (u, v) , (v, u) , $(1/u, v/u)$ is the CFT avatar of the anharmonic group of Theorem 4.4.

Remark 8.2 (Status of the CFT parallel, status **S/H**). The reduction (15)–(16) is textbook conformal field theory (**S**). Its reading as “the same modular hand-off, domain (Rung I) before function (Rung II), appearing in a second corner of physics” is the **H** overlay — evidence that the disciplined-domain construction is not an artefact of the amplitudes application but a recurring pattern.

9 Higher-Point Domains: Cluster Coordinates, Positivity, and Scattering Equations

9.1 Cluster $\mathcal{X}$ -coordinates

For n -point amplitudes with $n > 4$ the natural variables are the cluster $\mathcal{X}$ -coordinates of the relevant configuration space — for planar $\mathcal{N} = 4$ super-Yang–Mills, of $\text{Conf}_n(\mathbb{P}^3)$ in momentum-twistor space, equivalently of the Grassmannian $\text{Gr}(4, n)$. Cluster $\mathcal{X}$ -coordinates are ratios of Plücker brackets and are precisely the higher-rank generalization of the cross-ratio: they are the PGL-invariant, positive coordinates on the configuration space, and empirically they are the “symbol letters” out of which the amplitude’s transcendental function is spelled. For $n \leq 7$ the relevant Grassmannian cluster algebras are of finite type and produce a finite symbol alphabet; for $n = 8$ the cluster algebra is of infinite type, yet a finite symbol alphabet is still selected by the amplitude, a phenomenon that remains under active investigation.

9.2 The positive Grassmannian

The positive Grassmannian $\text{Gr}_{\geq 0}(k, n)$ — the locus where all Plücker coordinates are non-negative — is the combinatorial home of on-shell amplitude data. Its cell decomposition (by plabic graphs / the positroid stratification) organizes the residues and factorization channels of amplitudes, and the amplituhedron is the positive geometry whose canonical form is the integrand of $\mathcal{N} = 4$ amplitudes. For the purposes of Rung I we note only the structural fact: positivity is an extra piece of *discipline* on the domain of observable values — not every point of the complex configuration space is physical, and the positive part carves out the physical region.

9.3 The scattering equations

The Cachazo–He–Yuan scattering equations realize the moduli space $\mathcal{M}_{0,n}$ of n punctured spheres as *literally* the geometric home of massless n -point kinematics. The scattering equations

$$E_a = \sum_{b \neq a} \frac{k_a \cdot k_b}{\sigma_a - \sigma_b} = 0, \quad a = 1, \dots, n, \quad (17)$$

fix the puncture positions $\sigma_a \in \mathbb{P}^1$ in terms of the on-shell momenta k_a , defining a map from $\mathcal{M}_{0,n}$ to kinematic space. Under this map, factorization channels of the amplitude are in one-to-one correspondence with boundary divisors of $\overline{\mathcal{M}}_{0,n}$ (Theorem 5.1), and the scattering-equation map restricts to a diffeomorphism between the worldsheet associahedron in moduli space and the kinematic (ABHY) associahedron in kinematic space. This is the sharpest realization of the Rung I thesis: the moduli space of marked points on $\mathbb{P}^1$ is not an analogy for the kinematic domain — through the scattering equations, it *is* the kinematic domain.

Remark 9.1 (A forward pointer to Rung IV, status **H**). The boundary/residue tree of $\overline{\mathcal{M}}_{0,n}$ (Theorem 5.1) and the iterated-coaction tree of Rung IV are conjecturally the same combi-

natorial object. Proving (or disproving) this identification in general is, to our knowledge, open, and we return to it in the discussion. It is precisely the kind of statement whose *domain* side lives at Rung I and whose *decomposition* side lives at Rung IV — a concrete reason the two rungs must be kept modular yet composable.

10 Results

We collect the rung’s contributions.

1. **The observable arena is a field.** The kinematic domain of a physical process is a field $\mathbb{F}$ (Theorem 3.1); dimensionless observables live in its multiplicative group $\mathbb{F}^\times$ (Theorem 3.2).
2. **Cross-ratios are the frame-independent observables.** The cross-ratio is a PGL_2 -invariant (Theorem 4.2) and, via the anharmonic group (Theorem 4.4), its ordered ambiguity is exactly the six-element orbit (10).
3. **The four-point domain is the thrice-punctured line.** The cross-ratio identifies $\mathcal{M}_{0,4}$ with $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (Theorem 4.8), which is exactly the analytic domain of the polylogarithms of Rung II (Theorem 4.9).
4. **Higher domains have boundary structure.** The moduli spaces $\mathcal{M}_{0,n}$ carry simplicial coordinates (11) and Deligne–Mumford boundary divisors identified with factorization channels (Theorem 5.1, Theorem 5.2).
5. **Ratios are weight-one primitives.** $\mathbb{F}^\times \otimes \mathbb{Q}$ is canonically the weight-one part $\mathcal{L}_G(\mathbb{F})_1$ of the Goncharov Lie coalgebra (Theorem 6.1), so Rung I’s ratios are the irreducible informational units of the entire library.
6. **Kinematics is genuinely moduli of marked points.** Through the scattering equations (17), $\mathcal{M}_{0,n}$ is the geometric home of massless kinematics, with boundaries matching factorization channels (Section 9).

The *emergent property* of this rung is the conjunction of these facts: a **disciplined domain of observable values** — gauge-reduced by PGL_2 , organized by cross-ratios, compactified with physically meaningful boundaries, and identified (at four points) with precisely the punctured line the next rung needs.

11 Discussion

11.1 Why “disciplined”

The word “disciplined” in the emergent-property name is doing real work. A naive “domain of observable values” would be the full tuple space $(\mathbb{P}^1)^n$, or the space of all Mandelstam invariants — but these are riddled with redundancy (frame/gauge choices) and with unphysical regions. Rung I imposes three disciplines: (i) the PGL_2 -quotient removes the labelling

redundancy, leaving cross-ratios as the honest coordinates (Theorem 4.2, Theorem 4.8); (ii) positivity (the positive Grassmannian) carves out the physical region; and (iii) compactification organizes the degenerations into a clean boundary combinatorics (Theorem 5.1). The result is not merely a set of numbers but a *structured arena* — and it is structure, not raw values, that the later rungs act on.

11.2 Composition to Rung II

The modular composition $I \rightarrow II$ is concrete and provable, not metaphorical. Theorem 4.8 hands Rung II the domain $\mathbb{P}^1 \setminus \{0, 1, \infty\}$; Rung II populates that domain with the functions Li_n and their multiple generalizations, defined by iterated integration against the logarithmic forms $\frac{dt}{t}$ and $\frac{dt}{t-1}$ with singularities exactly at the three punctures. The punctures of Rung I become the branch points of Rung II; the boundary divisors of $\overline{\mathcal{M}}_{0,n}$ become the entries of the symbol; the anharmonic group of Theorem 4.4 becomes the functional-equation group of the polylogarithm. Nothing in this composition is arbitrary: each Rung II structure is forced by a Rung I structure.

11.3 Limitations

We are explicit about what is *not* claimed, in keeping with the S/H/P discipline.

- **The strongest results are status S as mathematics, H as universal physics.** Theorems 4.2, 4.8 and 6.1 are theorems of geometry and arithmetic. Their reading as statements about *all* physical observables is heuristic (**H**); by Theorem 2.1 that heuristic status propagates to any composite claim.
- **There is no single universal functor** $\text{Math} \rightarrow \text{Phys}$. Each rung is a local, domain-indexed construction. Rung I disciplines the values of one process at a time; it does not assert a global correspondence.
- **The higher-rank dictionary is only case-by-case.** A fully general correspondence between the cluster varieties $\text{Conf}_n(\mathbb{P}^k)$ for $k > 1$ and physical kinematic domains for higher-multiplicity amplitudes is established only in cases (hexagon, heptagon); the general statement is open.
- **The residue-tree/coaction-tree identification is conjectural.** Theorem 9.1 flags this as an open problem, not a theorem.

11.4 Open problems

1. A general dictionary between $\text{Conf}_n(\mathbb{P}^k)$ cluster varieties ($k > 1$) and physical kinematic domains for n -point higher-multiplicity amplitudes, beyond the hexagon/heptagon cases.
2. A dedicated worked comparison of the scattering-equation map (17) with the cross-ratio/cluster-coordinate description of $\mathcal{M}_{0,n}$, making explicit how the associahedron boundary combinatorics matches the simplicial coordinates (11).

3. The conjectural isomorphism between the boundary/residue tree of a positive geometry (Rung I) and the iterated-coaction tree of Rung IV — arguably the single most interesting open problem the library surfaces.

12 Conclusion

We have built the ground floor of a modular library whose thesis is that a Goncharov-style Lie coalgebra is a bookkeeping system for how observables decompose into irreducible informational constituents. At this rung the observable’s *domain* is constructed: a field $\mathbb{F}$ of kinematic values, its multiplicative group $\mathbb{F}^\times$ of dimensionless ratios, the cross-ratio as the universal PGL_2 -invariant, and the moduli spaces $\mathcal{M}_{0,n}$ as the arena of physically distinct configurations. The two load-bearing theorems — Möbius invariance (Theorem 4.2) and the identification $\mathcal{M}_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$ (Theorem 4.8) — establish that this arena is not arbitrary but is exactly the disciplined, gauge-reduced, punctured domain that the next rung’s special functions require. The forward pointer $\mathbb{F}^\times \otimes \mathbb{Q} \cong \mathcal{L}_G(\mathbb{F})_1$ (Theorem 6.1) shows further that these raw ratios are the irreducible weight-one primitives of the entire coalgebraic hierarchy. The emergent property — **a disciplined domain of observable values** — is thus both an endpoint (of Rung I’s construction) and a beginning (of the ladder). In the synthesis paper this ground floor closes into a loop: the real number a regulator finally produces at Rung V is itself a new element of Rung I’s $\mathbb{F}^\times$, a measured ratio one could feed back into a new kinematic configuration. Measurement, in this picture, is not an exit from the mathematical structure but a re-entry into its own ground floor.

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